Invertible Sobolev functions: counterexamples and applications to nonlinear elasticity

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Tesis doctoral dirigida por Daniel Faraco Hurtado y Carlos Mora Corral *a Irina.*

Porque fueron, somos.
Porque somos, serán.

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Contents

Introducción: resumen y conclusiones

Esta tesis trata de funciones Sobolev invertibles. En los Capítulos 1 y 2 estudiaremos homeomorfismos Sobolev con mal comportamiento, mientras que en el Capítulo 3 estudiaremos funciones Sobolev invertibles con buen comportamiento.

Para ser más precisos, en los Capítulos 1 y 2 construimos homeomorfismos Sobolev $u : \Omega \to \mathbb{R}^n$ para los cuales $\det Du = 0$ y, además, los menores de Du a partir de cierto orden son cero. Aquí Ω es un conjunto abierto de $\mathbb{R}^n$. En la raíz de este comportamiento patológico está la no satisfacción de ninguna de las siguientes propiedades deseables que toda función Sobolev $u : \Omega \to \mathbb{R}^n$ puede poseer:

a) La condición de Luzin N.

b) $\operatorname{Det} Du = \det Du$.

De hecho, las condiciones a) y b) están relacionadas, como explicaremos más adelante.

Diremos que $u : \Omega \to \mathbb{R}^n$ satisface la condición de Luzin N si para todo $S \subset \Omega$ tal que $\mathscr{L}^n(S) = 0$ se satisface $\mathscr{L}^n(u(S)) = 0$. Diremos que u satisface la condición de Luzin N^{-1} cuando $\mathscr{L}^n(u(S)) = 0$ implica $\mathscr{L}^n(S) = 0$.

Con la expresión $\det Du$ denotamos al determinante de Du, que esta definido c.t.p. suponiendo que u es una función Sobolev, mientras que $\operatorname{Det} Du$ es el determinante distribucional de u, que para $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ está definido como la distribución en Ω dada por

$$\langle \operatorname{Det} Du, \phi \rangle := -\frac{1}{n} \int_\Omega u(x) \cdot (\operatorname{cof} Du(x) D\phi(x))\, dx, \quad \phi \in C_c^\infty(\Omega),$$

cuando esta integral esté bien definida, i.e., cuando $(\operatorname{adj} Du) u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Esto ocurre, por ejemplo, si $\operatorname{cof} Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$ y $u \in L^{q'}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ para algún $q \in [1, \infty]$. Con $\langle \cdot, \cdot \rangle$ indicamos el producto de dualidad entre una distribución y una función C_c^∞.

Si para todo $K \subset\subset \Omega$ tenemos

$$\sup \left\{ \langle \operatorname{Det} Du, \phi \rangle : \phi \in C_c^\infty(\Omega),\, \operatorname{supp} \phi \subset K,\, \|\phi\|_{L^\infty} \leqslant 1 \right\} < \infty,$$

entonces $\operatorname{Det} Du$ puede extenderse de manera única a una medida de Radon en Ω, e identificamos $\operatorname{Det} Du$ con esa medida. Si, además, $\det Du \in L^1_{\mathrm{loc}}(\Omega)$ y la medida $\operatorname{Det} Du$ es igual a

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$\det Du\mathscr{L}^n|_\Omega$, entonces simplemente escribimos $\operatorname{Det} Du = \det Du$. Entonces, una forma más explícita de decir $\operatorname{Det} Du = \det Du$ es la igualdad

$$-\frac{1}{n}\int_\Omega u(x)\cdot(\operatorname{cof} Du(x) D\phi(x))\,dx = \int_\Omega \det Du(x)\phi(x)\,dx \quad \text{para todal } \phi \in C_c^\infty(\Omega). \tag{0.0.1}$$

La igualdad (0.0.1) es cierta para funciones $u \in C^2(\Omega,\mathbb{R}^n)$, como puede verse fácilmente integrando por partes, usando la fórmula de Laplace para el desarrollo por columnas del determinante y la igualdad $\operatorname{div}\operatorname{cof} Du = 0$ (llamada identidad de Piola). Averiguar la regularidad Sobolev mínima de u para la cual se satisface b) ha sido un intenso campo de investigación, como revisaremos el Capítulo 3.

El hecho de que no se satisfaga la condición a) es muy irreal desde el el punto de vista físico, ya que esto significaría que u crea materia de la nada. Además, si esa condición no se cumple, tampoco se cumple la fórmula del cambio de variables. De hecho, para una función Sobolev, tenemos que la condición de Luzin se satisface si y solo si la fórmula del cambio de variables también lo hace (ver, e.g., [72]).

Si no se satisface la condición b), el determinante puntual $\det Du$ proporciona muy poca información de las propiedades geométricas de u. Es más, la propiedad b) significa que la función u no presenta *cavitación* (formación de agujeros), ver [123, 143, 31, 78, 16]. Por lo tanto, si no se satisface b), no sólo puede aparecer cavitación, sino que además fenómenos patológicos pueden presentarse en la forma en que el material externo rellene la cavidad, o que las cavidades formadas tengan volumen cero en la configuración de referencia, y se concentren en un conjunto de Cantor en la configuración deformada; ver, e.g., [123, Section 11].

El interés por homeomorfismos Sobolev viene históricamente de dos puntos de vista:

- Teoría geométrica de funciones, esto es, la generalización a $\mathbb{R}^n$ de la teoría de funciones analíticas conformes de una variable compleja, particularmente las propiedades geométricas y de teoría de funciones, teniendo como resultado la extensa teoría de funciones cuasiconformes y la ecuación de Beltrami compleja. En los últimos veinte años, se ha descubierto que muchas de las propiedades de las funciones cuasiconformes y cuasirregulares también las tienen una clase de funciones más extensa, las así llamadas funciones de distorsión finita. Ver, por ejemplo, las monografías [67, 140, 94, 86].

- Elasticidad no lineal, que estudia deformaciones $u : \Omega \to \mathbb{R}^n$ con $n = 3$ que satisfacen la ecuación del movimiento de Cauchy y cuyos equilibrios estables típicamente minimizan la energía elástica, que está dada por una integral de una función de energía almacenada (ver, por ejemplo, la monografía [29]). El interés por las funciones Sobolev invertibles en elasticidad no lineal empezó con Ball [11]. En elasticidad no lineal, no se requiere que la deformación sea un homeomorfismo pero tiene que ser invertible, en un sentido que tiene que ser bien definido, para evitar la interpenetración de la materia. Esta noción no es obvia en absoluto y, debido a ello, ha habido numerosas enfoques diferentes.

En esta tesis trataremos los dos puntos de vista. Para ser precisos, en los Capítulos 1 y 2 adoptaremos el punto de vista de la teoría geométrica de funciones, y en el Capítulo 3, el de la elasticidad no lineal.

Las funciones Sobolev u del Capítulo 2 que satisfacen $\det Du = 0$ c.t.p. pueden ser construidas, además, para que sean homeomorfismos. Es sabido, [135], que si $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ es un homeomorfismo, entonces u satisface la condición de Luzin. Sin embargo, para funciones Sobolev $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ con $p < n$, el hecho de que u sea un homeomorfismo no lleva a una regularidad mejor que la típica de una función de $W^{1,p}$.

El estudio de homeomorfismos Sobolev que satisfacen o no la condición de Luzin, en relación a su regularidad Sobolev, se inició en [135], donde Resetnjak probó que los homeomorfismos en $W^{1,n}$ satisfacen la condición de Luzin. Previamente, Besicovitch [18] dio un ejemplo de una función $\phi \in W^{1,2}(\Omega, \mathbb{R}^3)$ con $\Omega \subset \mathbb{R}^2$ y $\mathscr{L}^3(\phi(\Omega)) > 0$; ver también [118]. En [132] y [133], Ponomarev contruyó un ejemplo de un homeomorfismo $u : [0,1]^n \to [0,1]^n$ de clase $W^{1,p}$ para todo $p < n$ para el cual falla la condición de Luzin. En [139] Resetnjak construyó un ejemplo de una función continua de clase $W^{1,n}(\mathbb{R}^n)$ que no satisface la condición de Luzin cuando $n = 2$. En [149], Väisälä extendió este resultado a todo $n \geqslant 2$. Hajlasz probó en [72] que si una función es aproximadamente diferenciable c.t.p. (por ejemplo, una función Sobolev de $W^{1,1}_{\text{loc}}$) entonces puede ser redefinida en un conjunto de medida cero para satisfacer tanto la condición de Luzin N como la fórmula del cambio de variables. De hecho, los resultados de Hajlasz fueron, en ese tiempo, más o menos conocidos en la comunidad que trabaja en espacios de Sobolev; ver [123, Section 2], que muestra cómo probar los resultados de Hajlasz usando teoremas clásicos de Federer [59] y Morrey [118]. Más tarde, Malý probó en [108] que dada $u \in W^{1,n}$, la fórmula del cambio de variables se satisface para u en el conjunto donde es aproximadamente continua Hölder; es más, el conjunto excepcional donde u no es aproximadamente continua Hölder tiene dimensión de Hausdorff cero. En [114], Martio y Ziemer dieron condiciones suficientes para que una función $u \in W^{1,n}$ cuyo jacobiano no cambiara de signo satisficiera la condición de Luzin; este trabajo fue extendido por Malý y Martio [110], donde quitaron la condición sobre el jacobiano y probaron que era suficiente con ser abierto y continuo. Una lista de condiciones suficientes para que una función continua en $W^{1,n}$ satisfaga la condición de Luzin puede ser también encontrada en [109]; están incluidas ser un homeomorfismo, tener jacobiano positivo, o ser abierta.

Ejemplos de funciones que no satisfacen la condición de Luzin son homeomorfismos Sobolev u cuyo determinante jacobiano es cero en casi todo punto, dado que si la satisficieran, entonces la fórmula del cambio de variables se cumpliría para ellos, y, por tanto

$$0 = \int_{\Omega} \det Du \, dx = \mathscr{L}^n(\Omega),$$

lo que contradice el hecho de que u es un homeomorfismo. En la literatura muchos autores han construido funciones u con $\det Du = 0$ c.t.p., con diferentes técnicas. En los siguientes parágrafos resumiremos algunos de ellos.

Alberti y Ambrosio dieron en [2] un ejemplo de una función Hölder $u \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ para todo $p < 2$ tal que $\det Du = 0$ c.t.p.; es más, u es el gradiente de una función convexa.

En [82], Hencl prueba que existe un homeomorfismo u en $W^{1,p}((0,1)^n, (0,1)^n)$, $1 \leqslant p < n$, cuyo determinante jacobiano Ju es igual a cero c.t.p. En [97] se prueba que una función Sobolev u tal que

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Omega} |Du|^{n-\varepsilon} = 0, \tag{0.0.2}$$

satisface la condición de Luzin, mientras que la construción de [24] mejora la de [82] para probar que existe un homeomorfismo $u \in W^{1,1}((0,1)^n, \mathbb{R}^n)$ tal que $Ju = 0$ en casi todo punto y Du está en el espacio grande de Lebesgue (*grand Lebesgue space*) $L^{n)}$, i.e.,

$$\sup_{0<\varepsilon\leqslant n-1} \varepsilon \int_{(0,1)^n} |Du|^{n-\varepsilon} < \infty.$$

La construción de Hencl [82] ha sido desarrollada en [50] para construir un homeomorfismo bi-Sobolev u con $Ju = 0$ y $Ju^{-1} = 0$ en casi todo punto.

Černý en [25] construye un homeomorfismo Sobolev con derivada de rango bajo y un homeomorfismo bi-Sobolev cuya derivada y la derivada de la inversa tienen rango bajo. Sin embargo, la integrabilidad de las derivadas está lejos de ser óptima.

Todas estas construcciones se basaban en una cuidadosa construcción explícita y en un proceso al límite para obtener un conjunto de Cantor donde la parte singular de la medida $\operatorname{Det} Du$ (o la parte singular de los menores distribucionales de orden más pequeño) estuviera soportada.

En [56], [129] y en esta tesis, usamos laminados y el método de integración convexa para obtener un homeomorfismo Sobolev cuya derivada tiene rango bajo y un homeomorfismo bi-Sobolev con su derivada y la derivada de la inversa de rango bajo con la integrabilidad óptima en la escala de los espacios de Sobolev. En [106] los autores construyen una función con características similares a la construida en [56], independientemente y casi simultáneamente.

La técnica de integración convexa empieza con el teorema de Nash-Kuiper, que dice que dada una inmersión f de una variedad de Riemann M^m de dimensión m a $\mathbb{R}^n$ con $n \geqslant m+1$ y dado $\varepsilon > 0$ existe una inmersión C^1, f_ε de M^m a $\mathbb{R}^n$ tal que $|f - f_\varepsilon| < \varepsilon$. Este teorema fue probado primero por Nash [127] para $n \geqslant m+2$ y luego mejorado por Kuiper [103] a su actual enunciado.

La condición $\det Du = 0$ c.t.p. puede verse como una inclusión en derivadas parciales, a saber $Du \in \mathbb{R}^{n\times n} \setminus GL(n)$ donde $GL(n)$ es el conjunto de matrices $n \times n$ invertibles. El estudio de inclusiones de la forma

$$Du \in K \text{ c.t.p.} \tag{0.0.3}$$

con $K \subset \mathbb{R}^n$ o $\mathbb{R}^{n\times n}$ un conjunto dado es crucial en areas de la física y la ingeniería para entender problemas como la minimización de una energía no convexa o la microestructura cristalina [13]. La energía a minimizar típicamente es de la forma

$$\int_\Omega W(Du)\,dx, \tag{0.0.4}$$

con W siendo no cuasiconvexo, $W \geqslant 0$ y $W^{-1}(\{0\}) = K$. El hecho de que W no es cuasiconvexo da una pista de que los minimizadores pueden no existir. Por otro lado, si u satisface (0.0.3) entonces obviamente minimiza (0.0.4). Una solución trivial a (0.0.3) es la función afín $u(x) = Ax + b$ con $A \in K$ y $b \in \mathbb{R}$ o $\mathbb{R}^n$, pero puede que no satisfaga las condiciones de frontera del problema. Por tanto, una cuestión fundamental concerniente a (0.0.3) es la existencia de soluciones Du no constantes. La posibilidad de solucionar (0.0.3) para Du no constante es llamado el problema exacto.

Si (0.0.3) no admite soluciones con Du no constantes, todavía nos podemos preguntar por la existencia de una sucesión $\{u_j\}_{j\in\mathbb{N}}$ de soluciones aproximadas, en el sentido que

$$\operatorname{dist}(Du_j, K) \to 0 \quad \text{en medida.} \tag{0.0.5}$$

El problema (0.0.5) es llamado el problema aproximado, y su cuestión fundamental es la existencia de esa sucesión $\{u_j\}_{j\in\mathbb{N}}$ con Du_j no convergiendo a una constante $A \in K$. Dicho de otro modo, si toda solución para (0.0.5) satisface que Du_j converge a una constante $A \in K$, decimos que el problema aproximado solo admite la solución trivial. Si ese es el caso, uno puede explorar el problema cuantitativo, i.e., si la desigualdad

$$\min_{A\in K} \int_\Omega |Du(x) - A|^p \, dx \lesssim \int_\Omega \operatorname{dist}(Du(x), K)^p \, dx \tag{0.0.6}$$

es válida. En el caso que $K \subset \mathbb{R}^{n\times n}$ tenga conexiones de rango uno, i.e., existen $A, B \in K$ tales que $\operatorname{rank}(A - B) = 1$, estos problemas son triviales. Esto es porque, en tal caso, podemos encontrar vectores distintos de cero $a, n \in \mathbb{R}^n$ tal que $A - B = a \otimes n$, y, por tanto, las funciones lineales correspondiendo a A y B coinciden a lo largo de todo un hiperplano de $\mathbb{R}^n$. Por tanto, tales dominios en los que la función afín tiene gradiente A puede tocar los dominios en los que el gradiente es igual a B a lo largo de este hiperplano sin violar la continuidad. Entonces es posible construir una función Lipschitz en un dominio conexo que usa precisamente los gradientes A y B. Por lo tanto, para los problemas (0.0.3), (0.0.5) y (0.0.6) asumimos que no hay conexiones de rango uno.

Dacorogna y Marcellini [41] estudiaron el caso escalar (i.e., cuando $K \subset \mathbb{R}^n$) del problema exacto para funciones Lipschitz.

Cuando $K = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$ la inclusión diferencial (0.0.3), o sus variantes (0.0.5) o (0.0.6), son conocidos como el problema de m matrices.

El caso más simple de este problema es $m = 2$, que fue estudiado por Ball y James en [13]. Šverák, [155], prueba, para el caso $m = 3$, que, si suponemos que no hay conexiones de rango uno y u soluciona (0.0.3), entonces Du es constante en casi todo punto (ver también [157, 5, 58]). También demostró que si u_j son Lipschitz y satisfacen (0.0.5) entonces $Du_j \to$ constante en medida. Este comportamiento, que también se cumple para $m = 2$, no se cumple para $m \geqslant 4$ como fue probado, para el problema aproximado, por Tartar [148] que usó "configuraciones T_4" de matrices 2×2, $A_1, \ldots, A_4$. Estas configuraciones T_4 fueron primero consideradas por Scheffer en [142], y después usadas por Aumann y Hart [8] (ejemplos similares fueron descubiertos por Casadio, [23], y Nesi y Milton [128] en diferentes contextos). Para el problema exacto, Chlebík y Kirchheim probaron en [28] que cualquier función Lipschitz usando cuatro gradientes sin conexiones de rango uno dos a dos era necesariamente afín. Faraco y Székelyhidi [57] probaron que el ejemplo de Tartar era la única construcción para la cual la propiedad (0.0.5) no implicaba que Du_j converge a una constante. En [100] Kirchheim y Preiss dieron una configuración de cinco matrices soportada en el conjunto de matrices simétricas 2×2 sin conexiones de rango uno que admite una solución no afín para el problema exacto. Es más, esta configuración es estable bajo pequeñas perturbaciones. Ver [64, 27, 36, 44, 14] para algunos resultados sobre el problema cuantitativo y sus aplicaciones a elasticidad no lineal.

Un problema relevante relacionado con este y con aplicaciones a la elasticidad no lineal es aquel en que, debido a la invariancia de la función de energía almacenada, consideramos conjuntos invariantes respecto a $SO(n)$. Este problema es llamado el problema de multipozos. Šverák [159] estudió el problema de dos pozos en dimensión dos, y después, con Müller [125], usaron los laminados y los métodos de Gromov [69] para avanzar en el problema. En [48], Dolzmann, Kirchheim, Müller y Šverák mejoraron el entendimiento del problema de dos pozos en dimensión tres. Para ver más acerca de estos problemas y su importancia consultar [121] y [99].

En [126], Müller y Šverák adaptarón las ideas de Gromov al ámbito de los laminados para solucionar (0.0.3) para K compacto. Por lo tanto, las soluciones eran Lipschitz. Usaron el método de resolver inclusiones diferenciales y una configuración de Tartar T_4 para proporcionar ejemplos de soluciones Lipschitz no diferenciables en ningún punto de la ecuación de Euler-Lagrange $\operatorname{div} DW(Du) = 0$ correspondiente a (0.0.4), donde Ω es un disco en dimensión dos, $u : \Omega \to \mathbb{R}^2$, y W es una función suave en $\mathbb{R}^{2\times 2}$ que es fuertemente cuasiconvexa con derivadas segundas $D^2 W$ uniformemente acotadas. Puesto que, por los resultados de Evans [52], los minimizadores absolutos de I son suaves fuera de una subconjunto cerrado de Ω de medida cero (esto es cierto incluso para los minimizadores locales, de acuerdo a un resultado de Kristensen y Taheri [102]), los ejemplos demuestran una gran diferencia entre la regularidad de las soluciones débiles y los minimizadores. De hecho, Scheffer había probado una versión de este resultado con W rango-uno convexo en vez de cuasiconvexo usando configuraciones T_4. Recordando el contraejemplo de Šverák [158] que demuestra que rango-uno convexidad no implica cuasiconvexidad, y el rol central jugado por la cuasicovexidad en el Cálculo de Variaciones, podemos ver la importancia de la extensión de ese resultado a W cuasiconvexa. Székelyhidi [147] extendió este resultado a W estrictamente policonvexa.

Todos los resultados mencionados arriba son para funciones Lipschitz, i.e., para K acotado. Fue Faraco en [54] quien, inspirado por una sugerencia de Milton [115], inventó los laminados escalera y usó los métodos de integración convexa en conjuntos no acotados para ver que el umbral para la integrabilidad de la derivada de soluciones de ecuaciones isótropas en el plano de la forma

$$\operatorname{div}(\rho D u) = 0 \quad \text{en } Q$$

es $\frac{2K}{K-1}$; en la última ecuación Q es un cubo en $\mathbb{R}^2$, $u \in W^{1,2}(Q, \mathbb{R})$ y $\rho \in L^\infty(Q, [\frac{1}{K}, K])$. Esta clase de laminados ha resultado ser extremadamente útil en muchos problemas diversos. En [34] los autores construyen un contraejemplo a la desigualdad de Korn en L^1. En [6] usaron estos laminados para construir, en dimensión dos, soluciones débiles con integrabilidad crítica, tanto para ecuaciones isótropas como para ecuaciones no en forma divergente. Estos ejemplos muestran que la teoría general de L^p, desarrollada en [4], [7] y [105], no puede ser extendida sin una restricción adicional en el rango esencial de los coeficientes. Otra aplicación de estos laminados fue obtener cotas inferiores para normas L^p de integrales singulares, [20]; esto fue desarrollado después por Bañuelos y Osękowski en un contexto probabilístico, [9]. En [56] usamos estos laminados para construir el homeomorfismo Sobolev con derivada de rango bajo mencionado arriba, como será explicado en la Sección 2.2 del Capítulo 2.

En [129] el autor es capaz de construir una sucesión de laminados escaleras cuyo soporte converge a matrices de rango bajo, y el soporte de la inversa también converge a matrices de rango bajo. Hasta donde nosotros sabemos, esta es la primera vez que alguien trabaja con una sucesión de laminados y de sus inversas al mismo tiempo. Esto será explicado en la Sección 2.3

del Capítulo 2.

En todos los ejemplos mencionados arriba, la condición de Luzin es violada y el determinante distribucional $\operatorname{Det} Du$ no es absolutamente continuo con respecto a la medida de Lebesgue, así que ninguna de las condiciones a)-b) de arriba se satisfacen. De hecho, las condiciones a)-b) admiten versiones de dimensiones más bajas, a saber, la condición de Luzin N en planos, y que los menores distribucionales sean iguales a los menores puntuales. En muchos de los ejemplos mencionados arriba (y, de hecho, en los del Capítulo 2) la versiones de dimensiones más bajas de a)-b) también fallan, pero no trataremos este asunto en esta tesis.

En la mayoría de casos (ver, e.g., [78]) $\operatorname{Det} Du$, que es en principio solo una distribución, es de hecho una medida. Es más, usando tanto la descomposición de Lebesgue como el teorema de Radon-Nikodym, el determinante distribucional puede expresarse como

$$\operatorname{Det} Du = f\,dx + \mu^s \tag{0.0.7}$$

donde $f \in L^1_{\text{loc}}(\Omega)$ y μ^s es una medida singular con respecto a la medida de Lebesgue. De hecho, en la mayoría de los casos ([78], [120]) $f = \det Du$. Recientemente, fue probado [49] que, para una función continua en $W^{1,n-1}(\Omega,\mathbb{R}^n)$ con $\det Du \in L^1(\Omega)$, la validez de la condición de Luzin implica que el determinante distribucional es igual al determinante puntual.

La descomposición (0.0.7) explica porqué el determinante puntual dice tan poco de las propiedades geométricas de u: porque estamos perdiendo la medida singular μ^s. Usualmente μ^s está soportado en un conjunto de Cantor, y este es el caso de [132] y muchos otros, pero véase [89] para un sorprendente ejemplo donde $\mu^s = \mathscr{H}^1$ en un segmento en dimensión 2.

En este trabajo no analizaremos el determinante distribucional, pero sería interesante explorar un método general de calcular μ^s cuando la función u está dada por la construcción de laminados. En algunos de los ejemplos, Du no tiene integrabilidad suficiente para poder definir $\operatorname{Det} Du$, por lo que se tendrían que calcular los menores distribucionales de rango más bajo.

En el Capítulo 3 adoptamos el punto de vista de la elasticidad no lineal. Dejamos de requerir que las deformaciones sean continuas. Esto es importante al hacer modelos ya que muchas deformaciones realistas presentan discontinuidades (como aquellas que corresponden al fenómeno de la cavitación, pero otras discontinuidades menos drásticas también son posibles). Dado que la deformación es Sobolev pero no necesariamente continua, solo está definida en casi todo punto (de hecho, excepto en un conjunto de p-capacidad cero, ver [53] o [164]), por lo que unas definiciones precisas de *invertible* e *inyectiva* son necesarias.

En su artículo pionero [11], Ball probó un resultado que garantiza que las deformaciones u en el espacio $W^{1,p}(\Omega,\mathbb{R}^n)$ con $p \geqslant n$ y $\det Du > 0$ satisfacen

$$\mathscr{L}^n\left(\left\{y \in u(\Omega) : \operatorname{Card} u^{-1}(y) > 1\right\}\right) = 0.$$

Esto fue, en la práctica, la primera definición de invertibilidad en este contexto. Otra posible definición es que u sea inyectiva c.t.p., i.e., existe un conjunto $\Omega_0 \subset \Omega$ de medida total tal que $u|_{\Omega_0}$ es inyectiva. Esta fue la definición considerada en [123]. Claramente, bajo las condiciones de Luzin N and N^{-1}, ambas definiciones son equivalentes.

Fueron Müller y Spector [123] quienes notaron que la inyectividad c.t.p. no era suficiente para prevenir la interpenetración de la materia (ver [123, Section 11], [124, Section 5] y [78, Section 7] para algunos ejemplos patológicos). La condición que la previene es llamada INV y, establece que casi toda esfera S en Ω es impenetrable, en el sentido que el material del interior de S va a material al interior de $u(S)$ y el material fuera de S va a material fuera $u(S)$.

Más estudios en invertibilidad han sido hechos en [31], [143], [76], [77], [78] y [79]. Estudios en invertibilidad local en el contexto de elasticidad no lineal han sido hechos en [62] y [16]. En [16] también se trata el tema de preservar la orientación, en particular, bajo qué condiciones $\det Du > 0$ c.t.p. implica que la orientación se preserva; ver, e.g. [78, 93] para funciones con $\det Du > 0$ que revierten la orientación.

El punto de partida del Capítulo 3 es la clase de funciones $\mathscr{A}_p$ definida en [16] como el conjunto de todas las funciones $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ $(p > n-1)$ tales que $\det Du > 0$ y que no crean nueva superficie (como cavidades). Los autores fueron capaces de probar que muchas de las propiedades de $W^{1,p}$ con $p > n$ se transladan a la clase $\mathscr{A}_p$, de la misma forma que Šverák [156] había probado que se trasladaban a $\mathscr{A}_{p,q}$ con $q \geqslant \frac{p}{p-1}$ y en [122] que se trasladaban a $\mathscr{A}_{p,q}$ con $q \geqslant \frac{n}{n-1}$. Esta clase $\mathscr{A}_{p,q}$ consiste en las funciones $u \in W^{1,p}$ tal que $\operatorname{cof} Du \in L^q$ y $\det Du > 0$ c.t.p. En el Capítulo 3 probaremos primero que todas las aproximaciones de invertibilidad usadas hasta ahora en elasticidad no lineal son equivalentes en la clase $\mathscr{A}_p$. También las generalizamos al rango de exponentes de $\mathscr{A}_p$.

En el Capítulo 3 también vemos un resultado de relajación en elasticidad no lineal en $\mathscr{A}_p$. La palabra *relajación* tiene un significado preciso en el Cálculo de Variaciones; se refiere a la envoltura semicontinua inferior, i.e., el mayor funcional semicontinuo inferiormente (en la topología adecuada) por debajo de uno dado. Es un resultado clásico de Young [163] que la relajación de

$$\int_\Omega W(u)dx \quad \text{es} \quad \int_\Omega W^c(u)dx$$

donde W^c es la *convexificación* de W, i.e., la mayor función convexa por debajo de W. Exposiciones modernas de este hecho pueden ser encontradas, e.g., en [51, 21, 63, 40].

Es también bien conocido [39] que la relajación de un funcional del tipo $\int_\Omega W(Du)dx$ es $\int_\Omega W^{qc}(Du)dx$, donde W^{qc}, la *cuasiconvexificación* de W, es la mayor función cuaxiconvexa por debajo de W. Sin embargo, ni este último resultado ni sus numerosas generalizaciones (ver, e.g., [17, 70, 71, 40, 160, 145, 146, 111]) cumplen las condiciones de crecimiento en elasticidad no lineal, en las cuales a la función de energía almacenada $W : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ se le requiere satisfacer

$$W(F) = \infty \text{ si } \det F \leqslant 0 \quad \text{y} \quad W(F) \to \infty \text{ cuando } \det F \to 0, \tag{0.0.8}$$

para evitar que se revierta la orientación.

Recientemente, Conti y Dolzmann [33] establecieron el primer resultado de relajación compatible con que la energía almacenada satisfaga (0.0.8). Vieron que la relajación está dada precisamente por

$$\int_\Omega W^{qc}(Du)dx.$$

Lo probaron para deformaciones en $W^{1,p}$ con $p \geqslant n$. También supusieron que W^{qc} es policonvexo para obtener la semicontinuidad inferior, dado que generalmente los teoremas de semicontinuidad inferior bajo (0.0.8) han sido hechos bajo policonvexidad (e.g. [12]) pero no bajo cuaxiconvexidad. En el Capítulo 3 generalizamos este resultado para cubrir la clase $\mathscr{A}_p$.

También tratamos con energías de la forma

$$\int_\Omega W(Du,\vec{n}(u))\,dx + \int_{u(\Omega)} |D\vec{n}(y)|^2\,dy. \tag{0.0.9}$$

Este tipo de energías aparecen en [15], [47] y [161] para modelar elastómeros nemáticos pero pueden ser de utilidad en otros contextos (ver [16]). Los elastómeros nemáticos son un tipo de elastómeros de cristales líquidos, que son una clase de material que combina las propiedades de los cristales líquidos con las de los sólidos parecidos al caucho, cuya estructura interna está formada por una red de cadenas de polímeros entrelazadas. En estas cadenas, unidades de monómeros rígidos alargados son incorporadas o unidas lateralmente. Si el orden de estas cadenas es uniaxial y el grado del orden está fijado, su orden orientacional está descrito por un campo director $\vec{n}$ de norma 1 definido en la configuración deformada; este campo describe la dirección de alargamiento de las moléculas en $u(x)$. Este campo director es la clave para entender el comportamiento anisotrópo. El primer término de la energía (0.0.9) es la energía mecánica, que une la energía elástica de la deformación con el campo director. El segundo término penaliza la no uniformidad espacial de los vectores directores. Ambas forman la energía del par deformación-orientación $(u,\vec{n})$. En [16] fue probada la existencia de minimizadores de (0.0.9) bajo la suposición de que W es policonvexo es su primera variable. En el Capítulo 3 vemos que si W no es ni siquiera cuasiconvexo, la relajación de (0.0.9) en la clase $\mathscr{A}_p$ es

$$\int_\Omega W^{qc}(Du,\vec{n}(u))\,dx + \int_{u(\Omega)} |D\vec{n}(y)|^2\,dy,$$

donde W^{qc} es la cuasiconvexificación de W con respecto a la primera. La principal hipótesis es, como en [33], que W^{qc} es policonvexo.

Los Capítulos 1 y 2 son parte de [56] y [129], mientras que el Capítulo 3 es parte de un artículo en preparación.

La estructura de la tesis es la siguiente. En el Capítulo 1 definimos el concepto de laminado y mostramos cómo construir una función cuya derivada esté cerca de un laminado dado. En el Capítulo 2 usamos el capítulo anterior para construir un homeomorfismo Sobolev y uno bi-Sobolev cuyas derivadas tienen rango bajo; en el caso de bi-Sobolev la derivada de la inversa también tiene rango bajo. También probamos que la integrabilidad de estos homeomorfismos es óptima estableciendo algunos resultados que relacionan la integrabilidad y el rango de la derivada de una función Sobolev. Finalmente, en el Capítulo 3 probamos que la relajación de una energía de un modelo de elastómeros nemáticos en la clase $\mathscr{A}_p$ es la cuasiconvexificación.

Introduction: summary and conclusions

This thesis deals with invertible Sobolev maps. In Chapters 1 and 2 we will study Sobolev homeomorphisms that are badly behaved, while in Chapter 3 we will study invertible Sobolev maps that are well behaved.

To be more precise, in Chapters 1 and 2 we construct Sobolev homeomorphisms $u:\Omega\to\mathbb{R}^n$ for which $\det Du = 0$ and, moreover, the minors of Du from some order are all zero. Here Ω is an open set of $\mathbb{R}^n$. At the root of this pathological behaviour is the non-satisfaction of any of the following desirable properties that a Sobolev map $u:\Omega\to\mathbb{R}^n$ may possess:

a) Luzin's condition N.

b) $\operatorname{Det} Du = \det Du$.

In fact, conditions a) and b) are related, as we will explain later.

We say that $u:\Omega\to\mathbb{R}^n$ satisfies Luzin's condition N if for all $S\subset\Omega$ such that $\mathscr{L}^n(S)=0$, there holds $\mathscr{L}^n(u(S))=0$. We say that u satisfies Luzin's condition N^{-1} when $\mathscr{L}^n(u(S))=0$ implies $\mathscr{L}^n(S)=0$.

With the expression $\det Du$ we denote the determinant of Du, which is defined a.e. provided that u is a Sobolev map, while $\operatorname{Det} Du$ is the distributional determinant of u, which for $u\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is defined as the distribution on Ω given by

$$\langle \operatorname{Det} Du,\phi\rangle := -\frac{1}{n}\int_\Omega u(x)\cdot(\operatorname{cof} Du(x) D\phi(x))\,dx, \quad \phi\in C^\infty_c(\Omega),$$

whenever this integral is well defined, i.e., when $(\operatorname{adj} Du)u \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. This happens, for example, if $\operatorname{cof} Du\in L^q_{\mathrm{loc}}(\Omega,\mathbb{R}^{n\times n})$ and $u\in L^{q'}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ for some $q\in[1,\infty]$. With $\langle\cdot,\cdot\rangle$ we indicate the duality product between a distribution and a C^∞_c function.

If for all $K\subset\subset\Omega$ we have

$$\sup\left\{\langle \operatorname{Det} Du,\phi\rangle : \phi\in C^\infty_c(\Omega),\ \operatorname{supp}\phi\subset K,\ \|\phi\|_{L^\infty}\leqslant 1\right\}<\infty,$$

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then $\operatorname{Det} Du$ can be extended uniquely to a Radon measure in Ω, and we identify $\operatorname{Det} Du$ with that measure. If, in addition, $\det Du \in L^1_{\mathrm{loc}}(\Omega)$ and the measure $\operatorname{Det} Du$ equals $\det Du \mathscr{L}^n|_\Omega$, then we simply write $\operatorname{Det} Du = \det Du$. Therefore, a more explicit way of saying that $\operatorname{Det} Du = \det Du$ is the equality

$$-\frac{1}{n}\int_\Omega u(x)\cdot(\operatorname{cof} Du(x) D\phi(x))\,dx = \int_\Omega \det Du(x)\phi(x)\,dx \quad \text{for all } \phi \in C_c^\infty(\Omega). \tag{0.0.10}$$

Equality (0.0.10) is true for $u \in C^2(\Omega,\mathbb{R}^n)$, as can be easily seen with an integration by parts, Laplace's formula for the development of the determinant in terms of the columns of the matrix, and the equality $\operatorname{div}\operatorname{cof} Du = 0$ (called Piola's identity). Ascertaining the minimal Sobolev regularity of u for which b) holds has been an intense field of research, as we will review in Chapter 3.

Condition a) not holding is very unrealistic from the physical point of view, because that would mean that u creates matter from nothing. In addition, if it is violated then the change of variables formula does not hold. In fact, for a Sobolev map, we have that Luzin's condition is satisfied if and only if the change of variables hold (see, e.g., [72]).

If condition b) is violated, the pointwise determinant $\det Du$ provides very little information about the geometric properties of u. Moreover, the property b) means that the map u does not exhibit *cavitation* (formation of voids), see [123, 143, 31, 78, 16]. So if b) is violated, not only cavitation can appear, but also, pathological phenomena can be present in the form that some material from outside fills the cavity, or that the cavities formed have zero volume in the reference configuration, and in the deformed configuration they concentrate on a Cantor set; see, e.g., [123, Section 11].

The interest for Sobolev homeomorphisms comes historically from two different points of view:

- Geometric function theory, that is, the generalization to $\mathbb{R}^n$ of the theory of conformal analytic functions of one complex variable, particularly the geometric and function-theoretic properties, having as a result the wide theory of quasiconformal mappings and the complex Beltrami equation. In the last twenty years, it has been discovered that many of the properties of quasiconformal and quasiregular mappings still hold for wider class of mappings, the so called mappings of finite distortion. See, for example, the monographs [67, 140, 94, 86].

- Nonlinear elasticity, which studies deformations $u : \Omega \to \mathbb{R}^n$ with $n = 3$ that satisfies Cauchy's equation of motion and whose stable equilibria typically minimize the elastic energy, which is given as an integral of a stored-energy function (see, for example, the monograph [29]). The interest for Sobolev invertible maps in nonlinear elasticity was started by Ball [11]. In nonlinear elasticity, the deformation is not required to be a homeomorphism but it has to be invertible, in a sense that has to be defined properly, so as to avoid interpenetration of matter. This notion is not obvious at all and, accordingly, there have been many different approaches.

In this thesis we work with both points of view. To be precise, in Chapters 1 and 2 we adopt the geometric function theory viewpoint and in Chapter 3 the nonlinear elasticity viewpoint.

The Sobolev maps u of Chapter 2 satisfying $\det Du = 0$ a.e. can be constructed, in addition, to be homeomorphisms. It is known, [135], that if $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ is a homeomorphism, then u satisfies Luzin's condition. However, for Sobolev maps $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p < n$, the fact that u is a homeomorphism does not lead to a better regularity that a typical $W^{1,p}$ function.

The study of Sobolev homeomorphisms satisfying or not Luzin's condition, according to its Sobolev regularity, started in [135], where Resetnjak proved that homeomorphisms in $W^{1,n}$ satisfy Luzin's condition. Previously, Besicovitch [18] provided an example of a function $\phi \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $\Omega \subset \mathbb{R}^2$ and $\mathscr{L}^3(\phi(\Omega)) > 0$; see also [118]. In [132] and [133], Ponomarev constructed an example of a homeomorphism $u : [0,1]^n \to [0,1]^n$ of class $W^{1,p}$ for all $p < n$ for which Luzin's condition fails. In [139] Resetnjak constructed an example of a continuous mapping of class $W^{1,n}(\mathbb{R}^n)$ that does not satisfy Luzin's condition when $n = 2$. In [149], Väisälä extended this result to any $n \geqslant 2$. Hajlasz proved in [72] that if a function is approximately differentiable a.e. (for example, a Sobolev $W^{1,1}_{\text{loc}}$ map) then it can be redefined in a set of measure zero to satisfy both Luzin's condition N and the change of variables formula. In fact, Hajlasz' results were, by that time, more or less known by the community working in Sobolev spaces; see [123, Section 2], which shows how to prove Hajlasz' results by using classical theorems by Federer [59] and Morrey [118]. Later on, Malý proved in [108] that given $u \in W^{1,n}$, the change of variables formula holds for u in the set where u is approximately Hölder continuous; moreover, the exceptional set where u is not approximately Hölder continuous has Hausdorff dimension zero. In [114], Martio and Ziemer give sufficient conditions for a mapping $u \in W^{1,n}$ whose Jacobian does not change its sign to satisfy Luzin's condition; this work was extended by Malý and Martio [110], where they removed the condition on the Jacobian and proved that it was enough to be open and continuous. A list of sufficient conditions for a continuous mapping in $W^{1,n}$ to satisfy Luzin's condition can be also found in [109]; they include being a homeomorphism, having positive Jacobian, or being open.

Examples of functions that do not satisfy Luzin's condition are Sobolev homeomorphisms u whose Jacobian determinants are equal to zero almost everywhere, since if they satisfied it, then the change of variables formula would hold for them, and, hence

$$0 = \int_\Omega \det Du\, dx = \mathscr{L}^n(\Omega),$$

which contradicts the fact that u is a homeomorphism. In the literature many authors had constructed mappings u with $\det Du = 0$ a.e., with different techniques. In the following paragraphs we summarize some of them.

Alberti and Ambrosio provided in [2] an example of a Hölder continuous function $u \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < 2$ such that $\det Du = 0$ a.e.; moreover u is the gradient of a convex function.

In [82], Hencl proves that there exists a homeomorphism u in $W^{1,p}((0,1)^n, (0,1)^n)$, $1 \leqslant p < n$, whose Jacobian determinant Ju equals zero a.e. In [97] it is proved that if a Sobolev map u is such that

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |Du|^{n-\varepsilon} = 0, \tag{0.0.11}$$

then u satisfies Luzin's condition, whereas the construction of [24] elaborates on that of [82] to show that there exists a homeomorphism $u \in W^{1,1}((0,1)^n, \mathbb{R}^n)$ such that $Ju = 0$ almost everywhere and Du is in the grand Lebesgue space $L^{n)}$, i.e.,

$$\sup_{0<\varepsilon\leqslant n-1} \varepsilon \int_{(0,1)^n} |Du|^{n-\varepsilon} < \infty.$$

The construction of Hencl [82] has been further developed in [50] to construct bi-Sobolev homeomorphisms u with $Ju = 0$ and $Ju^{-1} = 0$ almost everywhere.

Černý in [25] constructs a Sobolev homeomorphism with derivative of low rank and a bi-Sobolev homeomorphism whose derivative and the derivative of the inverse have low rank. However, the integrability of the derivatives is far for being sharp.

All those constructions were based on a careful explicit construction and a limit process to obtain a Cantor set where the singular part of the measure $\operatorname{Det} Du$ (or else the singular part of the distributional minors of lower order) is supported.

In [56], [129] and in this thesis, we use, instead, laminates and the method of convex integration to obtain a Sobolev homeomorphism with derivative of low rank and a bi-Sobolev homeomorphism with its derivative and the derivative of the inverse of low rank with the sharpest integrability in the scale of Sobolev spaces. In [106] the authors construct a function with similar characteristics of the one constructed in [56], independently and almost simultaneously.

The technique of convex integration starts with the Nash-Kuiper theorem, which states that given an immersion f from a Riemannian manifold M^m of dimension m to $\mathbb{R}^n$ with $n \geqslant m+1$ and given $\varepsilon > 0$ there is a C^1 immersion f_ε from M^m to $\mathbb{R}^n$ such that $|f - f_\varepsilon| < \varepsilon$. This theorem was proven first by Nash [127] for $n \geqslant m+2$ and then sharpened by Kuiper [103] to the current statement.

The condition $\det Du = 0$ a.e. can be seen as a partial differential inclusion, namely $Du \in \mathbb{R}^{n\times n} \setminus GL(n)$ where $GL(n)$ is the set of invertible $n \times n$ matrices. The study of inclusions of the form

$$Du \in K \text{ a.e.} \tag{0.0.12}$$

with $K \subset \mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ a given set, is crucial in areas as physics and engineering in order to understand problems like the minimization of a nonconvex energy or crystal microstructure [13]. The energy to minimize is typically of the form

$$\int_\Omega W(Du)\,dx \tag{0.0.13}$$

with W being non-quasiconvex, $W \geqslant 0$ and $W^{-1}(\{0\}) = K$. The fact that W is not quasiconvex gives a hint that minimizers may not exist. On the other hand, if u satisfies (0.0.12) then it obviously minimizes (0.0.13). A trivial solution to (0.0.12) is the affine map $u(x) = Ax + b$ with $A \in K$ and $b \in \mathbb{R}$ or $\mathbb{R}^n$, but it may not satisfy the boundary conditions of the problem. Thus, a fundamental question regarding (0.0.12) is the existence of solutions with Du non-constant. The possibility of solving (0.0.12) for Du non-constant is called the exact problem.

If (0.0.12) does not admit solutions with Du non-constant, one can still ask for the existence of a sequence $\{u_j\}_{j\in\mathbb{N}}$ of approximate solutions, in the sense that

$$\operatorname{dist}(Du_j, K) \to 0 \quad \text{in measure.} \tag{0.0.14}$$

Problem (0.0.14) is called the approximate problem, and its fundamental question is the existence of such sequences $\{u_j\}_{j\in\mathbb{N}}$ with Du_j not converging to a constant $A \in K$. Otherwise, if any sequence solving (0.0.14) satisfies that Du_j converges to a constant $A \in K$, we say that the approximate problem only admits the trivial solution. If this is the case, one may explore the quantitative problem, i.e., whether the inequality

$$\min_{A\in K}\int_\Omega |Du(x)-A|^p\,dx \lesssim \int_\Omega \operatorname{dist}(Du(x),K)^p\,dx \tag{0.0.15}$$

is valid. In the case that $K\subset\mathbb{R}^{n\times n}$ has rank-one connections, i.e., there exist $A,B\in K$ such that $\operatorname{rank}(A-B)=1$, these problems are trivial. That is because, in such a case, we can find nonzero vectors $a,n\in\mathbb{R}^n$ such that $A-B=a\otimes n$, and, hence, the linear mappings corresponding to A and B agree along a whole hyperplane of $\mathbb{R}^n$. Therefore, such domains on which an affine map has gradient A can touch such domains on which the gradient equals B along this hyperplane without violating the continuity. Then it is possible to construct a Lipschitz map on a connected domain that uses precisely the gradients A and B. Thus, for the problems (0.0.12), (0.0.14) and (0.0.15) one assumes that has no rank-one connections.

Dacorogna and Marcellini [41] studied the scalar case (i.e., when $K\subset\mathbb{R}^n$) of the exact problem for Lipschitz functions.

When $K=\{A_1,\dots,A_m\}\subset\mathbb{R}^{n\times n}$ the differential inclusion (0.0.12), or else its variants (0.0.14) or (0.0.15), are known as the m-matrices problem.

The simplest case of this problems is $m=2$, which was studied by Ball and James in [13]. Šverák, [155], proves, for the case $m=3$, that, if we suppose that there are not one-rank connections and u solves (0.0.12), then Du is constant almost everywhere (see also [157, 5, 58]). He also showed that if u_j Lipschitz satisfy (0.0.14) then $Du_j\to$ constant in measure. This behavior, that also holds for $m=2$, does not hold for $m\geqslant 4$ as it is shown, for the approximate problem, by Tartar [148] who used "T_4-configurations" of 2×2 matrices $A_1,\dots,A_4$. Those T_4 configurations were first considered by Scheffer in [142], and later used by Aumann and Hart [8] (similar examples were discovered by Casadio, [23], and Nesi and Milton [128] in different contexts). For the exact problem, it was proved by Chlebík and Kirchheim in [28] that any Lipschitz function using four pairwise not rank-one connected gradients is necessarily affine. Faraco and Székelyhidi [57] proved that the example of Tartar is the unique construction for which property (0.0.14) does not imply that Du_j converges to a constant. In [100] Kirchheim and Preiss gave a configuration of five matrices supported in the set of 2×2 symmetric matrices without one-rank connections that admit a non-affine solution for the exact problem. Moreover, this configuration is stable under small perturbations. See [64, 27, 36, 44, 14] for some results about the quantitative problem and its applications to nonlinear elasticity.

A relevant related problem with applications to nonlinear elasticity is when, due to the frame invariance of the stored energy function, we consider invariant sets with respect to $SO(n)$. This problem is called the multiwell problem. Šverák [159] studied the two-well problem in dimension two, and later, with Müller [125], used laminates and the methods of Gromov [69] to advance on the problem. In [48], Dolzman, Kirchheim, Müller and Šverák improved the understanding of the two-well problem in three dimensions. To see more about these problems and their importance consult [121] and [99].

In [126], Müller and Šverák adapted the ideas of Gromov to the laminates setting to solve (0.0.12) for K compact. Therefore, the solutions were Lipschitz. They used this method of solving differential inclusions and a T_4 Tartar's configuration to provide examples of nowhere differentiable Lipschitz solutions to the Euler-Lagrange equation $\operatorname{div} DW(Du) = 0$ corresponding to (0.0.13), where Ω is a disk in dimension two, $u : \Omega \to \mathbb{R}^2$, and W is a smooth function on the set $\mathbb{R}^{2\times 2}$ that is strongly quasiconvex with uniformly bounded second derivatives $D^2 W$. Since, by the result of Evans [52], absolute minimizers of I are smooth outside a closed subset of Ω of measure zero (this is even true for local minimizers, according to a result of Kristensen and Taheri [102]), the examples demonstrate a great difference between the regularity of weak solutions and that of minimizers. In fact, Scheffer had proved a version of this result with W rank-one convex instead of quasiconvex using T_4-configurations. Recalling the counterexample of Šverák [158] that rank-one convexity does not imply quasiconvexity, and the central role played by the quasiconvexity in the calculus of variations, we can see the importance of the extension of that result to quasiconvex W. Székelyhidi [147] extended this result to a strictly polyconvex W.

All the results mentioned above were for Lipschitz functions, i.e., for bounded K. It was Faraco in [54] who, inspired by a suggestion of Milton [115], invented the staircase laminates and used the method of convex integration to unbounded sets to show that the threshold for the integrability of the gradient of solutions to planar isotropic equations of the form

$$\operatorname{div}(\rho Du) = 0 \quad \text{in } Q$$

is $\frac{2K}{K-1}$; in the last equation Q is a cube in $\mathbb{R}^2$, $u \in W^{1,2}(Q,\mathbb{R})$ and $\rho \in L^\infty(Q,[\frac{1}{K},K])$. This kind of laminates has turned to be extremely useful in many diverse problems. In [34] the authors construct a counterexample to Korn's inequality in L^1. In [6] they used these laminates to construct, in dimension two, weak solutions with critical integrability properties, both to isotropic equations and to equations in non-divergence form. These examples show that the general L^p theory, developed in [4], [7] and [105], cannot be extended under any restriction on the essential range of the coefficients. Another application of these laminates was to obtain lower bounds for L^p norms of singular integrals, [20]; this was developed later by BaÃśuelos and Osękowski in a probabilistic context, [9]. In [56] we use these laminates to construct the Sobolev homeomorphisms with derivative of low rank mentioned above, as will be explained in Section 2.2 of Chapter 2.

In [129] the author is able to construct a sequence of staircase laminates whose support converges to matrices of low rank, and the support of the inverse laminates also converges to matrices of low rank. As far as we know, this is the first time that someone works with a sequence of laminates and their inverses at the same time. This will be explained in Section 2.3 of Chapter 2.

In all the examples mentioned above, Luzin's condition is violated and the distributional determinant $\operatorname{Det} Du$ is not absolutely continuous with respect to the Lebesgue measure, so none of the conditions a)-b) above holds. In fact, conditions a)-b) admit lower-dimensional versions, namely, the Luzin's condition N in planes, and the distributional minors being equal to the pointwise minors. In many examples above (and, in fact, in those of Chapter 2) the lower dimensional versions of a)-b) also fail, but we will not deal with this issue in this thesis.

In most cases (see, e.g., [78]) $\operatorname{Det} Du$, which is in principle only a distribution, is in fact a measure. Moreover, using the Lebesgue decomposition as well as the Radon-Nikodym theorem, the distributional determinant can be expressed as

$$\operatorname{Det} Du = f\,dx + \mu^s \tag{0.0.16}$$

where $f \in L^1_{\mathrm{loc}}(\Omega)$ and μ^s is a singular measure with respect to the Lebesgue measure. In fact, in most cases ([78], [120]) $f = \det Du$. Recently, it was proved [49] that for a continuous mapping in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ with $\det Du \in L^1(\Omega)$ the validity of Luzin's condition implies that the distributional determinant equals the pointwise determinant.

The decomposition (0.0.16) explains why the pointwise determinant says little about the geometric properties of u: because we are missing out the singular measure μ^s. Usually μ^s is supported in a Cantor set, and this is the case for [132] and many others, but see [89] for a surprising example where $\mu^s = \mathcal{H}^1$ in a line segment in dimension 2.

In this work we do not analyze the distributional determinant, but it will be interesting to explore a general method of calculating μ^s when the function u is given through the construction of laminates. In some of the examples, Du does not have enough integrability in order for $\operatorname{Det} Du$ to be defined, so one would have to calculate the distributional minors of lower rank.

In Chapter 3 we adopt the viewpoint of nonlinear elasticity. The deformations are no longer required to be continuous. This is important in the modelling since many realistic deformations present discontinuities (like those corresponding to the phenomenon of cavitation, but other less drastic discontinuities are also possible). Since the deformation is Sobolev but not necessarily continuous, it is only defined almost everywhere (in fact, up to a set of p-capacity zero, see [53] or [164]), so a proper definition of *invertible* and *injective* are necessary.

In his pioneering paper [11], Ball proved a result guaranteeing that deformations u in the space $W^{1,p}(\Omega,\mathbb{R}^n)$ with $p \geq n$ and $\det Du > 0$ satisfy

$$\mathcal{L}^n\left(\left\{y \in u(\Omega) : \operatorname{Card} u^{-1}(y) > 1\right\}\right) = 0.$$

This was, in practice, the first definition of invertibility in this context. Another possible definition is that u is injective a.e., i.e., there exists a set $\Omega_0 \subset \Omega$ of full measure such that $u|_{\Omega_0}$ is injective. This was the definition considered in [123]. Clearly, under Luzin's conditions N and N^{-1}, both definitions are equivalent.

It was Müller and Spector [123] who noticed that injectivity a.e. is not enough to prevent interpenetration of matter (see [123, Section 11], [124, Section 5] and [78, Section 7] for some pathological examples). The condition that prevents it was called INV and, roughly speaking, states that almost every sphere S in Ω is impenetrable, in the sense that material inside S goes to material inside $u(S)$ and material outside S goes to material outside $u(S)$.

Future studies of invertibility have been done in [31], [143], [76], [77], [78] and [79]. Studies of local invertibility in the context of nonlinear elasticity have been done in [62] and [16]. In [16] it is also addressed the issue of orientation preserving, in particular, under which conditions $\det Du > 0$ a.e. implies the preservation of orientation; see, e.g. [78, 93] for mappings with $\det Du > 0$ that reverse the orientation.

The starting point of Chapter 3 is the class of functions $\mathcal{A}_p$ defined in [16] as the set of all functions $u \in W^{1,p}(\Omega,\mathbb{R}^n)$ $(p > n-1)$ such that $\det Du > 0$ and they do not create new surface (like cavities). The authors were able to prove that many properties of $W^{1,p}$ with $p > n$ translate to the class $\mathscr{A}_p$, in the same way that Šverák [156] had proved that they translate to $\mathscr{A}_{p,q}$ with $q \geqslant \frac{p}{p-1}$ and in [122] that they translate to $\mathscr{A}_{p,q}$ with $q \geqslant \frac{n}{n-1}$. This class $\mathscr{A}_{p,q}$ consists of the functions $u \in W^{1,p}$ such that $\operatorname{cof} Du \in L^q$ and $\det Du > 0$ a.e. In Chapter 3 we first prove that all approaches of invertibility used so far in nonlinear elasticity are equivalent within the class $\mathscr{A}_p$. We also generalize them all to the range of exponents of $\mathscr{A}_p$.

In Chapter 3 we also show a relaxation result in nonlinear elasticity set in $\mathscr{A}_p$. The word *relaxation* has a precise meaning in Calculus of Variations; it refers to the lower semicontinuous envelope, i.e., the largest lower semicontinuous functional (in the appropriate topology) below a given one. It is a classical result going back to Young [163] that the relaxation of

$$\int_\Omega W(u)\,dx \quad \text{is} \quad \int_\Omega W^c(u)\,dx$$

where W^c is the *convexification* of W, i.e., the largest convex function below W. Modern expositions of this fact can be found, e.g., in [51, 21, 63, 40].

It is also well-known [39] that the relaxation of a functional of the type $\int_\Omega W(Du)\,dx$ is $\int_\Omega W^{qc}(Du)\,dx$, where W^{qc}, the *quasiconvexification* of W, is the largest quasiconvex function below W. However, neither this latter result nor its many generalizations (see, e.g., [17, 70, 71, 40, 160, 145, 146, 111]) meet the growth assumptions in nonlinear elasticity, in which the stored energy function $W : \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ is required to satisfy

$$W(F) = \infty \text{ if } \det F \leqslant 0 \quad \text{and} \quad W(F) \to \infty \text{ as } \det F \to 0, \tag{0.0.17}$$

so as to avoid orientation reversal.

Recently, Conti and Dolzmann [33] established the first result of relaxation compatible with a stored energy satisfying (0.0.17). They showed that the relaxation is given precisely by

$$\int_\Omega W^{qc}(Du)\,dx.$$

They proved it for $W^{1,p}$ deformations with $p \geqslant n$. They also supposed that W^{qc} is polyconvex to obtain the lower semicontinuity, since general theorems of lower semicontinuity under (0.0.17) have been done under polyconvexity (e.g. [12]) but not under quasiconvexity. In Chapter 3 we generalize their result to cover the class $\mathscr{A}_p$.

We also deal with energies of the form

$$\int_\Omega W(Du, \vec{n}(u))\,dx + \int_{u(\Omega)} |D\vec{n}(y)|^2\,dy. \tag{0.0.18}$$

This type of energies appear in [15], [47] and [161] to model nematic elastomers but they may be useful in other contexts (see [16]). Nematic elastomers are a type of liquid crystals elastomers, which are a kind of material that combines the properties of liquid crystals and rubber-like solids, whose inner structure is made by a network of cross-linked polymer chains. In those chains, elongated rigid monomer units are incorporated or attached sideways. If the order of those chains is uniaxial and the degree of the order is fixed, their orientational order is described by a director field $\vec{n}$ of norm 1 defined in the deformed configuration; it describes the direction of alignment of the molecules at $u(x)$. This vector field is the key to understand the anisotropic behaviour. The first term of the energy (0.0.18) is the mechanical energy, which couples the elastic energy of the deformation with the director field. The second term penalizes the spatial non-uniformity of directors. Both make up the energy of the pair deformation-orientation $(u, \vec{n})$. In [16] it was proved the existence of minimizers of (0.0.18) under the assumption that W is polyconvex in its first variable. In Chapter 3 we show that if W is not even quasiconvex, the relaxation of (0.0.18) in the class $\mathscr{A}_p$ is

$$\int_{\Omega} W^{qc}(Du, \vec{n}(u))\,dx + \int_{u(\Omega)} |D\vec{n}(y)|^2\,dy,$$

where W^{qc} is the quasiconvexification of W with respect to the first variable. The main assumption is, as in [33], that W^{qc} is polyconvex.

Chapters 1 and 2 are part of [56] and [129], while Chapter 3 is part of a paper in preparation.

The structure of the thesis is the following. In Chapter 1 we define the concept of laminate and show how to construct a function whose derivative is close to a given laminate. In Chapter 2 we use the previous chapter to construct a Sobolev and a bi-Sobolev homeomorphism whose derivatives have low rank; in the case of the bi-Sobolev the derivative of the inverse has also low rank. We also prove that the integrability of these homeomorphisms is sharp by establishing some results that correlate the integrability and the rank of the derivative a Sobolev map. Finally, in Chapter 3 we prove that the relaxation of the energy of a model of nemastic elastomers in the class $\mathscr{A}_p$ is the quasiconvexification.

Chapter 1

Laminates and their approximation by functions

In this chapter we will focus on explaining a method and giving the needed ingredients to solve partial differential inclusions of the form

$$Df \in F \text{ a.e.}$$

This is applied in Chapter 2 when we construct a Sobolev homeomorphism with derivative of low rank, and a bi-Sobolev homeomorphism with the derivatives of f and the inverse of f being of low rank.

The method to do this is to find a suitable family of laminates (see Definition 1.0.1) and then use Proposition 1.3.5 to find a sequence of homeomorphisms whose limit satisfies that its derivative is in the desired set, in our case, that will be the matrices with a given rank.

Hence the two principal components of this kind of constructions are Proposition 1.3.5, which, given a laminate, states that there exists a function close to that laminate; and the laminates themselves. The next definition introduces the concept of laminate of finite order [40, 131, 126, 99, 6].

Definition 1.0.1. *The family $\mathscr{L}(\mathbb{R}^{n\times n})$ of laminates of finite order is the smallest family of probability measures in $\mathbb{R}^{n\times n}$ with the properties:*

i) *$\mathscr{L}(\mathbb{R}^{n\times n})$ contains all the Dirac masses.*

ii) *If $\sum_{i=1}^N \lambda_i \delta_{A_i} \in \mathscr{L}(\mathbb{R}^{n\times n})$ and $A_N = \rho B + (1-\rho)C$, where $\rho \in [0,1]$ and $\operatorname{rank}(B-C) = 1$, then the probability measure*

$$\sum_{i=1}^{N-1} \lambda_i \delta_{A_i} + \lambda_N(\rho\delta_B + (1-\rho)\delta_C)$$

is also in $\mathscr{L}(\mathbb{R}^{n\times n})$.

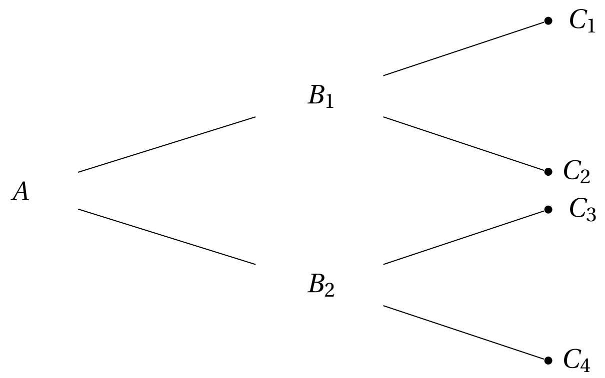
Note that any laminate of finite order is a convex combination of Dirac masses. Since in this work we will only use laminates of finite order, for simplicity they will be just called *laminates*.

The next lemma gives us a characterization of the laminates of finite order and will be used to prove Corollary 1.0.3

Lemma 1.0.2. *For every laminate of finite order $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ there exists a family $\{\nu_j\}_{j=1}^{N}$ of laminates of finite order, such that*

- $\nu_1 = \delta_{\overline{\nu}}$ *(where $\overline{\nu}$ is the barycenter of ν),*
- *for $j \in \{1,\dots,N-1\}$, ν_{j+1} is obtained from ν_j using once ii) of Definition 1.0.1,*
- $\nu_N = \nu$.

Lemma 1.0.2 has an elementary proof. It is illustrated in the following example. We have the splits



and $\nu = \sum_{i=1}^{4} \lambda_i \delta_{C_i}$. Then the laminates $\{\nu_i\}_{i=1}^{4}$ can be chosen to be the following:

$$\nu_4 = \nu,$$

$$\nu_3 = (\lambda_1 + \lambda_2)\delta_{B_1} + \lambda_3 \delta_{C_3} + \lambda_4 \delta_{C_4},$$

$$\nu_2 = (\lambda_1 + \lambda_2)\delta_{B_1} + (\lambda_3 + \lambda_4)\delta_{B_2},$$

$$\nu_1 = \delta_A.$$

From Lemma 1.0.2 we obtain the following corollary that will be used throughout Chapter 2 to prove that some measures are laminates.

Corollary 1.0.3. *Let $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ and $\{\nu_{A_i}\}_{i=1}^{N}$ be laminates of finite order such that*

$$\overline{\nu}_{A_i} = A_i \text{ with } A_i \text{ being all different.}$$

Then, the probability measure

$$\nu' = \sum_{i=1}^{N} \lambda_i \nu_{A_i} = \nu + \sum_{i=1}^{N} \nu(A_i)[\nu_{A_i} - \delta_{A_i}]$$

is also a laminate of finite order.

Next we define the inverse of a laminate. This concept is the key to control in Section 2.3 of Chapter 2 the derivative of the function and the derivative of the inverse at the same time.

Definition 1.0.4. *We define the inverse of a laminate* $\nu = \sum_{i=1}^{N} \lambda_i \delta_{A_i}$ *supported in the set of positive definite matrices as*

$$\nu^{-1} = \frac{1}{\det(\overline{\nu})} \sum_{i=1}^{N} \lambda_i \det(A_i) \delta_{A_i^{-1}}.$$

This definition, which seems to be new, arises from the fact that if $\sum_{i=1}^{N} \lambda_i \delta_{A_i}$ is a laminate supported in the set of positive definite matrices and f is a piecewise affine Sobolev homeomorphism satisfying

$$|E_i| = \lambda_i \text{ and } f_i(x) = A_i x \text{ for } x \in \partial E_i \text{ and } i \in \{1, \ldots, N\},$$

for

$$E_i = \{x \in \Omega : |Df(x) - A_i| < \delta\},$$

and some $\delta > 0$, then we get

$$|f(E_i)| = \int_{E_i} \det Df(x) dx = \int_{E_i} \det A_i dx = \lambda_i \det A_i \qquad \text{for } i \in \{1, \ldots, N\},$$

and hence, there exists $\delta' > 0$ such that f^{-1} satisfies

$$|\{y \in f(\Omega) : |Df^{-1}(y) - A_i^{-1}| < \delta'\}| = |f(E_i)| = \lambda_i \det A_i.$$

Although we will not use it, ν^{-1} is also a laminate; this can be shown using the fact that det is rank-one linear.

The outline of the chapter is as follows. In Section 1.1 we explain the notation of Chapters 1 and 2. In Section 1.2 we prove that if we paste Hölder continuous functions that coincide in the border, then, the result is also Hölder continuous. Finally in Section 1.3 we show how to construct a function whose derivative is close to a given laminate.

1.1 Notation of Chapters 1 and 2

We explain the general notation used throughout Chapters 1 and 2, most of which is standard.

In the whole thesis, Ω is an open, non-empty bounded set of $\mathbb{R}^n$.

We denote by $\mathbb{R}^{n\times n}$ the set of $n\times n$ matrices, by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ its subset of symmetric matrices, by Γ_+ its subset of symmetric positive semidefinite matrices, by $SO(n)\subset\mathbb{R}^{n\times n}$ the orthogonal matrices with determinant 1, and by I the identity matrix.

Given $A,B\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$ we write $A\leqslant B$ to denote that $B-A\in\Gamma_+$.

Given $A_i\in\mathbb{R}^{n\times n}$, the measure δ_{A_i} is the Dirac delta at A_i. The *barycenter* of the probability measure $\nu=\sum_{i=1}^N \alpha_i\delta_{A_i}$ is $\overline{\nu}=\sum_{i=1}^N \alpha_i A_i$.

Given $A\in\mathbb{R}^{n\times n}$, let $\sigma_1(A)\leqslant\cdots\leqslant\sigma_n(A)$ denote its singular values. If the matrix A is clear from the context, we will just indicate their singular values as $\sigma_1,\ldots,\sigma_n$. In fact, we will always deal with $A\in\Gamma_+$, so its eigenvalues coincide with its singular values. Its components are written $A_{\alpha,\beta}$ for $\alpha,\beta\in\{1,\ldots,n\}$. Its operator norm is denoted by $|A|$, which coincides with $\sigma_n(A)$. The norm of a $v\in\mathbb{R}^n$ is also denoted by $|v|$.

Given $a_1,\ldots,a_n\in\mathbb{R}$ the matrix $\operatorname{diag}(a_1,\ldots,a_n)\in\mathbb{R}^{n\times n}$ is the diagonal matrix with diagonal entries $a_1,\ldots,a_n$.

We will use the symbol $\lesssim$ when there exists a constant depending only on n, m, m_1 and m_2 such that the left hand side is less than or equal to the constant times the right hand side. Sometimes, the left hand side could be negative. Here, n is the dimension of the space, m is used in Section 2.2 to denote a number such that $\operatorname{rank} Df<m$, and m_1 and m_2 are used in Section 2.3 to denote numbers such that $\operatorname{rank} Df=m_1$ and $\operatorname{rank} Df^{-1}=m_2$.

Given a set $E\subset\mathbb{R}^n$, we denote its characteristic function by χ_E. We write $\operatorname{Card} E$ for the number of elements of E. When it is measurable, its Lebesgue measure is denoted by $|E|$ and we use $\mathscr{H}^m(E)$ for its Hausdorff measure of dimension m.

Given $a\in\mathbb{R}$, its integer part is denoted by $\lfloor a\rfloor$ and we denote by $\lceil a\rceil$ its ceiling function.

Given $E\subset\mathbb{R}^n$, $\alpha\in(0,1]$ and a function $f:E\to\mathbb{R}^n$, we denote the Hölder seminorm, supremum norm and Hölder norm, respectively, as

$$\begin{aligned}
|f|_{C^\alpha(E)} &:= \sup_{\substack{x_1,x_2\in E\\ x_1\neq x_2}} \frac{|f(x_2)-f(x_1)|}{|x_2-x_1|^\alpha}, \qquad \|f\|_{L^\infty(E)}=\sup_{x\in E}|f(x)|,\\
\|f\|_{C^\alpha(E)} &:= |f|_{C^\alpha(E)}+\|f\|_{L^\infty(E)}.
\end{aligned}$$

We will write $f\in C^\alpha(E,\mathbb{R}^n)$ when $\|f\|_{C^\alpha(E)}<\infty$. Note that, if f is continuous up to the boundary, the above norms and seminorms in E coincide with those in $\overline{E}$. In particular, we will identify $C^\alpha(E,\mathbb{R}^n)$ with $C^\alpha(\overline{E},\mathbb{R}^n)$, the set of Hölder functions of exponent α. Of course, if $\alpha=1$, they are Lipschitz.

The identity function is denoted by id and the Sobolev space from Ω to $\mathbb{R}^n$ is denoted, alternatively, by $W^{1,p}$, $W^{1,p}(\Omega)$ or $W^{1,p}(\Omega,\mathbb{R}^n)$.

Given $f:A\to\mathbb{R}^n$, where A is a subset of an m-dimensional affine space of $\mathbb{R}^n$, we say that it satisfies the m-dimensional Luzin condition, also known as condition N, if for every $E\subset A$ with $\mathscr{H}^m(E)=0$, then $\mathscr{H}^m(f(E))=0$.

For $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, we denote the Jacobian of f by Jf or $\det Df$ and, for $k \in \{1, \dots, n\}$, the k-dimensional Jacobian of f by $J_k f$, i.e., $J_k f$ is equal to $\left(\sum_M |M|^2\right)^{\frac{1}{2}}$, where the sum runs over all the minors of Df of order k.

We will say that a continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ is *piecewise affine* if there exists a countable family $\{\Omega_i\}_{i\in\mathbb{N}}$ of pairwise disjoint open subsets of Ω such that $f|_{\Omega_i}$ is affine for all $i \in \mathbb{N}$, and

$$\left| \Omega \setminus \bigcup_{i\in\mathbb{N}} \Omega_i \right| = 0.$$

Note that $\{\Omega_i\}_{i\in\mathbb{N}}$ need not be locally finite. Given $S \subset \mathbb{R}^{n\times n}$ a set of invertible matrices we denote by S^{-1} the set

$$\{A^{-1} : A \in S\}.$$

Given $\Omega, \Omega' \subset \mathbb{R}^n$ we denote its distance by $\operatorname{dist}(\Omega, \Omega')$; and we denote the diameter of Ω by $\operatorname{diam}(\Omega)$. In the case that Ω' consists of only one point, i.e., $\Omega' = \{x\}$ we will write $\operatorname{dist}(\Omega, x) = \operatorname{dist}(\Omega, \{x\})$.

Given $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ we denote by $B(\Omega, \varepsilon)$ the open set

$$\{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \varepsilon\}.$$

Given $x, y \in \mathbb{R}^n$, we denote their scalar product by $\langle x, y\rangle$.

1.2 Cutting and pasting Hölder homeomorphisms

In this section we will prove that if we modify a Hölder map (respectively a bi-Hölder homeomorphism) in some sets by cutting and pasting other Hölder maps (bi-Hölder homeomorphisms), the modified map is still a Hölder map (bi-Hölder homeomorphism).

First we show how to bound the C^α norm of a function in Ω with its C^α norms in a collection of subsets of Ω that covers Ω up to measure zero.

Lemma 1.2.1. *Let $\alpha \in (0,1]$, $\{\Omega_i\}_{i=1}^\infty \subset \Omega$ pairwise disjoint open sets such that $|\Omega \setminus \bigcup_{i=1}^\infty \Omega_i| = 0$, and $g : \overline{\Omega} \to \mathbb{R}^n$ such that $g(x) = 0$ for all $x \in \bigcup_{i=1}^\infty \partial\Omega_i$. Then*

$$\|g\|_{C^\alpha(\overline{\Omega})} \leqslant 2 \sup_{i\in\mathbb{N}} \|g\|_{C^\alpha(\overline{\Omega_i})}.$$

Proof. We assume that the right hand side of the last inequality is finite. Given $x, y \in \bigcup_{i=1}^\infty \overline{\Omega_i}$, let $i_0, i_1 \in \mathbb{N}$ be such that $x \in \overline{\Omega_{i_0}}$ and $y \in \overline{\Omega_{i_1}}$.

If $i_0 = i_1$, then $|g(x) - g(y)| \leqslant |g|_{C^\alpha(\overline{\Omega_{i_0}})} |x-y|^\alpha$, whereas if $i_0 \neq i_1$, we have $\Omega_{i_0} \cap \Omega_{i_1} = \emptyset$. Consider

$$\lambda_0 := \min\{\lambda \in [0,1] : x + \lambda(y-x) \in \partial\Omega_{i_0}\}, \quad \lambda_1 := \max\{\lambda \in [0,1] : x + \lambda(y-x) \in \partial\Omega_{i_1}\}$$

and let $x' = x + \lambda_0(y-x)$ and $y' = x + \lambda_1(y-x)$. Then, $g(x') = g(y') = 0$, and, therefore,

$$\begin{aligned} |g(x) - g(y)| &\leqslant |g(x) - g(x')| + |g(y') - g(y)| \leqslant |g|_{C^\alpha(\overline{\Omega_{i_0}})} |x - x'|^\alpha + |g|_{C^\alpha(\overline{\Omega_{i_1}})} |y' - y|^\alpha \\ &\leqslant \left(|g|_{C^\alpha(\overline{\Omega_{i_0}})} + |g|_{C^\alpha(\overline{\Omega_{i_1}})} \right) |x-y|^\alpha. \end{aligned}$$

On the other hand, $\|g\|_{L^\infty\left(\bigcup_{i=1}^\infty \overline{\Omega_i}\right)} = \sup_{i\in\mathbb{N}} \|g\|_{L^\infty\left(\overline{\Omega_i}\right)}$. Hence,

$$\|g\|_{C^\alpha(\bigcup_{i=1}^\infty \overline{\Omega_i})} \leqslant 2 \sup_{i\in\mathbb{N}} \|g\|_{C^\alpha(\overline{\Omega_i})}.$$

As $\bigcup_{i=1}^\infty \overline{\Omega_i}$ is dense in $\overline{\Omega}$, the required bound holds due to the uniform continuity. □

Lemma 1.2.2. *Let $f : \overline{\Omega} \to \mathbb{R}^n$ be a homeomorphism such that $f, f^{-1} \in C^\alpha(\Omega)$ for some $\alpha \in (0,1]$. Let $\{\omega_i\}_{i\in\mathbb{N}} \subset \Omega$ be a family of pairwise disjoint open sets, and for each $i \in \mathbb{N}$ let $g_i : \overline{\omega_i} \to f(\overline{\omega_i})$ be a homeomorphism such that $g_i = f$ on $\partial\omega_i$,*

$$\sup_{i\in\mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})} < \infty \quad \textit{and} \quad \sup_{i\in\mathbb{N}} \|f^{-1} - g_i^{-1}\|_{C^\alpha(f(\overline{\omega_i}))} < \infty.$$

Then, the function

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \omega_i, \\ g_i(x) & \text{if } x \in \omega_i \text{ for some } i \in \mathbb{N} \end{cases}$$

is a homeomorphism between $\overline{\Omega}$ and $f(\overline{\Omega})$ such that $\tilde{f}$ and $\tilde{f}^{-1}$ are C^α and

$$\|f - \tilde{f}\|_{C^\alpha(\overline{\Omega})} \leqslant 2 \sup_{i\in\mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})}, \qquad \|f^{-1} - \tilde{f}^{-1}\|_{C^\alpha(\overline{\Omega})} \leqslant 2 \sup_{i\in\mathbb{N}} \|f^{-1} - g_i^{-1}\|_{C^\alpha(f(\overline{\omega_i}))}.$$

Proof. Using that f and g_i are homeomorphisms we have that $f(\omega_i) = g_i(\omega_i)$, for each $i \in \mathbb{N}$. Thus, it is clear that the function

$$f(\overline{\Omega}) \ni y \longmapsto \begin{cases} f^{-1}(y) & \text{if } y \in f\left(\overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \omega_i\right), \\ g_i^{-1}(y) & \text{if } y \in f(\omega_i) \text{ for some } i \in \mathbb{N} \end{cases}$$

is the inverse of $\tilde{f}$.

Using Lemma 1.2.1, we obtain that

$$\|\tilde{f} - f\|_{C^\alpha(\overline{\bigcup_{i\in\mathbb{N}} \omega_i})} \leq 2 \sup_{i\in\mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})},$$
$$\|\tilde{f}^{-1} - f^{-1}\|_{C^\alpha(f(\overline{\bigcup_{i\in\mathbb{N}} \omega_i}))} \leq 2 \sup_{i\in\mathbb{N}} \|f^{-1} - g_i^{-1}\|_{C^\alpha(f(\overline{\omega_i}))}.$$

Moreover, if we call $F := \tilde{f} - f$, we have that $F = 0$ in $\overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \omega_i$. In order to show that F is C^α in $\overline{\Omega}$, given $x_1 \in \overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \omega_i$ and $x_2 \in \bigcup_{i\in\mathbb{N}} \omega_i$, we take $x_3 = x_1 + \lambda(x_2 - x_1)$ for some $\lambda \in [0,1]$ such that $x_3 \in \partial \bigcup_{i\in\mathbb{N}} \omega_i$. Then $F(x_1) = F(x_3) = 0$ and, hence,

$$|F(x_2) - F(x_1)| = |F(x_3) - F(x_2)| \leq \|F\|_{C^\alpha(\overline{\bigcup_{i\in\mathbb{N}} \omega_i})} |x_3 - x_2|^\alpha \leq \|F\|_{C^\alpha(\overline{\bigcup_{i\in\mathbb{N}} \omega_i})} |x_2 - x_1|^\alpha.$$

This shows that $F \in C^\alpha(\overline{\Omega}, \mathbb{R}^n)$. Analogously, $\tilde{f}^{-1} - f^{-1}$ is C^α in $f(\overline{\Omega})$ and the last bound of the statement also holds. In particular, $\tilde{f}$ and $\tilde{f}^{-1}$ are C^α, and, hence, $\tilde{f}$ is a homeomorphism between $\overline{\Omega}$ and $f(\overline{\Omega})$. □

Proceeding as in the proof of Lemma 1.2.2 we can obtain the next lemma.

Lemma 1.2.3. *Let $n, m \in \mathbb{N} \setminus \{0\}$, $f : \overline{\Omega} \to \mathbb{R}^m$ be a map such that f is C^α for some $\alpha \in (0,1]$. Let $\{\omega_i\}_{i\in\mathbb{N}} \subset \Omega$ be a family of pairwise disjoint open sets, and for each $i \in \mathbb{N}$ let $g_i : \overline{\omega_i} \to f(\overline{\omega_i})$ be such that $g_i = f$ on $\partial\omega_i$,*

$$\sup_{i\in\mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})} < \infty \quad .$$

Then, the function

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \omega_i, \\ g_i(x) & \text{if } x \in \omega_i \text{ for some } i \in \mathbb{N} \end{cases}$$

maps $\overline{\Omega}$ to $f(\overline{\Omega})$, is a C^α function and

$$\|f - \tilde{f}\|_{C^\alpha(\overline{\Omega})} \leq 2 \sup_{i\in\mathbb{N}} \|f - g_i\|_{C^\alpha(\overline{\omega_i})}.$$

1.3 Approximation of laminates by functions

In this section we will develop the tools that we will use in Chapter 2 to construct Sobolev and bi-Sobolev homeomorphisms with gradients of low rank. The results of this section state that given a finite laminate we can find a function whose derivative is close to that laminate. The main result of this section and the only one that we will use is Proposition 1.3.5, whose parts *(a)*, *(b)*, *(c)* and *(d)* are classical in the theory of laminates. For the sake of the completeness of the proof of Proposition 1.3.5 we include here Lemma 1.3.2 and Propositions 1.3.3 and 1.3.4 together with their proofs, which can be found in [99, Lemma 3.3 and Proposition 3.4] (the first two) and in [6, Lemma 2.1] (the last one). We will use Proposition 1.3.4 combined with the proof of [56, Proposition 4.1] to prove Proposition 1.3.5.

The next lemma states that if $f : \Omega \to \mathbb{R}^n$ is such that $Df \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then there exists $u : \Omega \to \mathbb{R}$ such that $f = Du$.

Lemma 1.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $B \supset \overline{\Omega}$ be a ball $f \in W^{1,1}(B, \mathbb{R}^n)$ such that $Df \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ a.e. Then there exists $u : \Omega \to \mathbb{R}$ in $W^{2,1}$ such that $f = \nabla u$ almost everywhere in Ω. Moreover, if $Df \in \Gamma_+$ a.e., then u is convex.*

Proof. Choose a family $\{\eta_\varepsilon\}_{\varepsilon>0}$ of standard mollifiers, and define $f_\varepsilon := f * \eta_\varepsilon$ in a ball $B_\varepsilon \subset B$ containing $\overline{\Omega}$. Then, $Df_\varepsilon(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ for all $x \in B_\varepsilon$. Moreover, if $Df \in \Gamma_+$ a.e., then $Df_\varepsilon(x) \in \Gamma_+$ for all $x \in B_\varepsilon$. This is because the sets $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ and Γ_+ are convex. Consequently, the differential 1-form $\alpha_\varepsilon := \sum_{i=1}^n f_\varepsilon^i \, dx_i$ defined in B_ε is closed, i.e., $d\alpha_\varepsilon = 0$, thanks to the symmetry of Df_ε. Here f_ε^i are the components of f_ε. By Poincaré's lemma, α_ε is exact, i.e., there exists a smooth function $u_\varepsilon : B_\varepsilon \to \mathbb{R}$ such that $du_\varepsilon = \alpha_\varepsilon$, so $\nabla u_\varepsilon = f_\varepsilon$. We can take u_ε such that $\int_\Omega u_\varepsilon = 0$. In the case $Df \in \Gamma_+$ a.e. the Hessian of u_ε is symmetric positive semidefinite, hence, u_ε is convex. Now, $f_\varepsilon \to f$ in $W^{1,1}(\Omega, \mathbb{R}^n)$ as $\varepsilon \to 0$. Thanks to the Poincaré inequality, there exists $u \in W^{2,1}(\Omega)$ such that $u_\varepsilon \to u$ in $W^{2,1}(\Omega)$. Therefore, $\nabla u = f$. Moreover, if $Df \in \Gamma_+$ a.e., then u is convex as a limit of convex functions. □

The existence of a function g (proved in the Lemma 1.3.2 below) that is at the same time C^1 and piecewise affine is quite surprising. In particular, it implies that the underlying affine decomposition of the domain of g is not locally finite.

Lemma 1.3.2. *[99, Lemma 3.3] Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $f \in C^1(\Omega, \mathbb{R}^n)$ such that $Df \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ everywhere. For any lower semicontinuous function $\varepsilon : \Omega \to (0, \infty)$ we can find a piecewise affine $g \in C^1$ with symmetric gradient satisfying $|Df(x) - Dg(x)| + |f(x) - g(x)| < \varepsilon(x)$ for all $x \in \Omega$.*

Proof. We can suppose that $f \in C^1(\overline{\Omega})$ and ε is a constant just by assuming $g = f$ and $Dg = Df$ on $\partial\Omega$ since the statement is local and lower semicontinuous functions attain their minimum in compact sets. Moreover, it is enough to prove the following.

(A) For each $\varepsilon > 0$ there exists an $\tilde{f} \in C^1(\overline{\Omega})$ with symmetric gradient and a open G such that $\overline{G} \subset \Omega$, $\tilde{f}|_G$ is locally affine, $|\partial G| = 0$, $|G| > \frac{|\Omega|}{2^{n+1}}$ and that $\|D\tilde{f} - Df\|_{L^\infty} + \|\tilde{f} - f\|_{L^\infty} < \varepsilon$ with $f = \tilde{f}$, $D\tilde{f} = Df$ on $\partial\Omega$.

Indeed, if (A) is true then we can construct a sequence of functions $f_k \in C^1(\overline{\Omega})$ and a sequence of open sets $G_k \subset G_{k+1} \subset \Omega$ such that

- $f_k = f$ and $Df_k = Df$ on $\partial\Omega$,
- $\|Df_{k+1} - Df_k\|_{L^\infty} + \|f_{k+1} - f_k\|_{L^\infty} < \frac{\varepsilon}{2^k}$,
- $|\Omega \setminus G_k| < (1-2^{-n-1})^k|\Omega|$, $|\partial G_k| = 0$ and $\overline{G}_k \subset \Omega$,
- $f_{k+1} = f_k$ on G_k, f_k being locally affine in G_k and $Df_k \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ everywhere.

To construct this sequence we apply (A) to $f_k|_{\Omega\setminus\overline{G_k}}$. Once the sequence $\{f_k\}$ is constructed, denote as g the limit of f_k (which, in fact, is the limit in the topology of $C^1(\overline{\Omega})$). We obtain that $g \in C^1(\overline{\Omega})$ has a symmetric gradient everywhere, $g = f$ and $Dg = Df$ on $\partial\Omega$, and it is locally affine in $\bigcup_k G_k$. Therefore, as this last set is of full measure, we obtain that g is piecewise affine in Ω.

To prove (A) let $\psi : \mathbb{R}^n \to [0,1]$ in C^∞, being zero in $B(0,\frac{3}{4})$ and with $\psi(x) = 1$ for $|x| > \frac{4}{5}$. Set $c_1 = \|D^2\psi\|_{L^\infty} + \|D\psi\|_{L^\infty} > 1$. Given $\varepsilon > 0$, let $\overline{B}_i = \overline{B}(x_i, r_i)$ be a family of N disjoint closed balls in Ω such that $\max_{x,y\in\overline{B}_i} |Df(x) - Df(y)| < \frac{\varepsilon}{16c_1}$, $r_i < 1$ and $\left|\bigcup_{i=1}^N \overline{B}_i\right| > \frac{|\Omega|}{2}$. Set $A_i = Df(x_i) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ for all $i \leqslant N$. Using that $Df - A_i$ is symmetric, Lemma 1.3.1 and $f \in C^1$, we get that for some $\delta > 0$ there exists $F_i \in C^2(B(x_i, r_i+\delta))$ with $DF_i(x) = f(x) - A_i x - f(x_i) + A_i x_i$ and $F_i(x_i) = 0$. Denote $\tilde{F}_i(x) = \psi\left(\frac{x-x_i}{r_i}\right)F_i(x)$. We use $\|F_i\|_{L^\infty(\overline{B}_i)} \leqslant r_i\|DF_i\|_{L^\infty(\overline{B}_i)}$ and $\|DF_i\|_{L^\infty(\overline{B}_i)} \leqslant r_i\|Df - A_i\|_{L^\infty(\overline{B}_i)}$ to obtain

$$\begin{aligned}\|D^2\tilde{F}_i\|_{L^\infty(\overline{B}_i)} &\leqslant c_1 r_i^{-2}\|F_i\|_{L^\infty(\overline{B}_i)} + 2c_1 r_i^{-1}\|DF_i\|_{L^\infty(\overline{B}_i)} + \|Df - A_i\|_{L^\infty(\overline{B}_i)} \leqslant 4c_1\|Df - A_i\|_{L^\infty(\overline{B}_i)}\\ &\leqslant 4c_1 \max_{x,y\in\overline{B}_i} |Df(x) - Df(y)| < \frac{\varepsilon}{4}.\end{aligned}$$

Hence, $\tilde{f}(x) := D\tilde{F}_i(x) + A_i x + f(x_i) - A_i x_i$ is in $C^1(\overline{B}_i)$, satisfies $\tilde{f}(x) = f(x)$ if $|x - x_i| > \frac{4}{5}r_i$ and, thanks to $D\tilde{F}_i(x) = 0$ if $|x - x_i| < \frac{3}{4}r_i$, we have that $\tilde{f}$ is affine in the open ball $B(x_i, \frac{3}{4}r_i)$. Moreover

$$\|D\tilde{f}_i - A_i\|_{L^\infty(\overline{B}_i)} \leqslant \|D^2\tilde{F}_i\|_{L^\infty(\overline{B}_i)} < \frac{\varepsilon}{4},$$

which implies $\|D\tilde{f}_i - Df\|_{L^\infty(\overline{B}_i)} < \frac{\varepsilon}{4} + \frac{\varepsilon}{16c_1} < \frac{\varepsilon}{2}$. We also have $\|\tilde{f}_i - f\|_{L^\infty(\overline{B}_i)} \leqslant r_i\|D\tilde{f}_i - Df\|_{L^\infty(\overline{B}_i)} < \frac{\varepsilon}{2}$. Therefore, if we define $\tilde{f} = \tilde{f}_i$ in B_i and $\tilde{f} = f$ elsewhere, and we set $G = \bigcup_{i=1}^N B(x_i, \frac{3}{4}r_i)$ we see that $\tilde{f}$ and G satisfy all the requirements of (A). □

Proposition 1.3.3. *[99, Proposition 3.4] Let $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ satisfy* $\operatorname{rank}(A - B) = 1$ *and let $C = \lambda A + (1-\lambda)B$ with $\lambda \in [0,1]$. Then, for every domain $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ there is a piecewise affine function $f : \Omega \to \mathbb{R}^n$ such that*

a) $f(x) = Cx$ if $x \in \partial\Omega$ and $\|f(x) - Cx\|_{L^\infty(\Omega)} < \varepsilon$,

b) $Df(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \cap B([A,B],\varepsilon)$ a.e. in Ω, and

c) $|\{x \in \Omega : Df(x) = A\}| > (1-\varepsilon)\lambda|\Omega|$ and $|\{x \in \Omega : Df(x) = B\}| > (1-\varepsilon)(1-\lambda)|\Omega|$.

Proof. Without loss of generality (by a translation, a rotation and a homothety) we can suppose that $C = 0$, $A = (1-\lambda)a \otimes a$ and $B = -\lambda a \otimes a$ for some $a \in \mathbb{R}^n$ of unit length.

Consider the cylinder $P = \{x + ta : t \in [0,1] \text{ and } x \in a^\perp \cap \overline{B}(0,1)\}$, where $a^\perp$ is the orthogonal space to a. Given $\varepsilon > 0$, fix $r \in (0,1)$ with $1 - r^{n-1} < \frac{\varepsilon}{2}$, and a C^∞-function $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi(s) = 1$ if $|s| \leqslant \frac{2r+1}{3}$ and $\varphi(s) = 0$ if $|s| \geqslant \frac{2+r}{3}$. Finally, we choose $H, h : \mathbb{R} \to \mathbb{R}$ Lipschitz such that $H(0) = 0$, $H' = h$ and h is 1-periodic with $h'(t) = 1 - \lambda - \chi_{[\frac{\lambda}{2}, 1-\frac{\lambda}{2}]}(t)$ for $t \in [0,1]$. Note that $h(\frac{1}{2}) = 0$ and that $h'(\frac{1}{2} + t) = h'(\frac{1}{2} - t)$ if $0 \leqslant t \leqslant \frac{1}{2}$ and hence, $h(\frac{1}{2} + t) = -h(\frac{1}{2} - t)$ for the same t. This shows that $\int_0^1 h = 0$, and therefore H is 1-periodic as well.

For an integer k large enough define the C^1 function

$$F_P(x) := \frac{1}{k^2} H(k\langle x, a\rangle)\varphi(|x - a\langle x, a\rangle|),$$

and the Lipschitz map

(1.3.1)
$$f_P(x) = DF_P(x) = \frac{1}{k} h(k\langle x, a\rangle)\varphi(|x - a\langle x, a\rangle|)a + \frac{1}{k^2} H(k\langle x, a\rangle)\varphi'(|x - a\langle x, a\rangle|)\frac{x - a\langle x, a\rangle}{|x - a\langle x, a\rangle|}.$$

Note that $f_P(x) = \frac{h(k\langle x,a\rangle)}{k} a$ if $x \in P$ and $|x - a\langle x, a\rangle| \leqslant r < \frac{2r+1}{3}$. So, we conclude $Df_P(x) = h'(k\langle x, a\rangle)a \otimes a \in \{A, B\}$ for almost every such x. This proves *c)* in the case $\Omega = P$. It is also clear that $f_P(x) = 0$ if $x \in P$ and $|x - a\langle x, a\rangle| > \frac{2+r}{3}$. Since $H(k) = h(k) = H(0) = h(0) = 0$ we infer that $f_P(x) = 0$ if $x \in \partial P$, hence *a)* is proved noting that we can choose k large enough to obtain $\|f_P\|_{L^\infty(P)} < \varepsilon$. Notice that Df_P exists and agrees with the second distributional derivative of F_P almost everywhere, and therefore it is a symmetric matrix. It remains to verify that $Df_P(x) \in B([A,B], \varepsilon)$ a.e. As $\varphi'(|x - a\langle x, a\rangle|)\frac{x - a\langle x,a\rangle}{|x - a\langle x,a\rangle|}$ and $\frac{H(k\langle x,a\rangle)}{k}$ are uniformly bounded in the Lipschitz norm for $k \in \mathbb{N}$, we can choose k large enough to obtain that the Lipschitz norm of the second summand in (1.3.1) is as small as we wish. Hence, we can bound it by $\frac{\varepsilon}{2}$. Similarly, if we differentiate the first summand in (1.3.1), then the term containing the derivative of φ is multiplied by $\frac{h(k\langle x,a\rangle)}{k}$, which is arbitrarily small, and contributes at most $\frac{\varepsilon}{2}$ to DF_p. Therefore, Df_P is up to an error of size ε equal to $h'(k\langle x, a\rangle)\varphi(|x - a\langle x, a\rangle|)a \otimes a \in [A, B]$, which establishes *b)*.

Since f_P is C^1 on each of the k-subcylinders $P_j = \{x \in P : k \cdot \langle x, a\rangle \in [j-1, j]\}$, $j = 1, \ldots, k$ we can apply Lemma 1.3.2 to each of the pieces $P_j^\circ \cap \{x : |x - a\langle x, a\rangle| > r\}$ to get the Lipschitz map $\tilde{f}_P$. To translate this function onto a general domain Ω, we fill Ω up to a set of measure zero with countably many mutually disjoint affine copies of P and we define f on each of them as the affine rescaled copy of $\tilde{f}_P$. □

The following proposition is the same as [6, Lemma 2.1] and parts *(i)-(iv)* are very similar to [126, Lemma 3.1], with the difference that in [126] they work with the C^0 norm, instead of the C^α norm.

Proposition 1.3.4. *[6, Lemma 2.1] Let $n, m \in \mathbb{N} \setminus \{0\}$, $A, B \in \mathbb{R}^{n\times m}$ with* $\operatorname{rank}(A - B) = 1$. *Consider $\lambda \in [0,1]$ and call $C := \lambda A + (1-\lambda)B$. Then, for every $\alpha \in (0,1)$, $0 < \delta < \frac{1}{2}|A - B|$ and every bounded open set $\Omega \subset \mathbb{R}^n$, there exists a piecewise affine Lipschitz map $f : \Omega \to C\Omega$ such that*

(i) $f(x) = Cx$ *for* $x \in \partial\Omega$,

(ii) $\|f(x) - Cx\|_{C^\alpha(\overline{\Omega})} < \delta$,

(iii) $|\{x \in \Omega : |Df(x) - A| < \delta\}| = \lambda|\Omega|$,

(iv) $|\{x \in \Omega : |Df(x) - B| < \delta\}| = (1-\lambda)|\Omega|$.

If we also have $m = n$ and $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, then the map f can be chosen so that

(v) $Df \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *a.e. in* Ω.

Proof. The structure of the proof is the following: first we will prove *(i)-(iv)* without supposing that $m = n$ and A, B symmetric, and then we will use Proposition 1.3.3 to construct a function that also satisfies *(v)*.

Without loss of generality, we can suppose $A - B = a \otimes e_n$ for some $a \in \mathbb{R}^n \setminus \{0\}$, where e_n is the last element of the canonical basis of $\mathbb{R}^n$. Define the function $s : (-\lambda, 1-\lambda) \to \mathbb{R}$ by

$$s(t) = \lambda(1-\lambda) + t\big((1-\lambda)\chi_{(-\lambda,0)}(t) - \lambda\chi_{(0,1-\lambda)}(t)\big) \tag{1.3.2}$$

and extend it by zero to $\mathbb{R}$. Then, s is continuous in $\mathbb{R}$. Given $0 < \delta' < \frac{1}{n|a|}\delta$, define $w : \Omega \to \mathbb{R}$ as

$$w(x) = \delta'\left[s\left(\frac{x_n}{\delta'}\right) - \sum_{i=1}^{n-1}|x_i|\right].$$

Let $f_0(x) = Cx + w(x)a$. It is easy to see that f_0 satisfies *(i)*, (iii) and *(iv)* in $\Omega_0 = \{x \in \mathbb{R}^n : w(x) > 0\}$. To see (ii) observe that the function s and the function s' defined as $s'(x) = -\sum_{i=1}^{n-1}|x_i|$ are Lipschitz continuous. Hence, for δ' small enough we have that the Lipschitz norm of $\delta' s'$ goes to zero, and also does the Hölder norm. We also have that for every $\alpha < 1$ the Hölder norm of $\delta' s\left(\frac{x}{\delta'}\right)$ goes to zero when δ' goes to zero. Therefore (ii) is proved. For an arbitrary domain Ω, we cover Ω up to measure zero by small copies of Ω_0 and we define f as the rescaled copy of f_0 in Ω_0. Since f is Lipschitz, the Hölder norm of $f(x) - Cx$ decreases when we make the copies of Ω_0 smaller, and thanks to f_0 satisfying *(i)-(iv)* in Ω_0 we have that f satisfies *(i)-(iv)* in Ω.

To obtain the function that satisfies *(i)-(v)*, under the assumption $A, B \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, we start with the piecewise affine function $g : \Omega \to \mathbb{R}^n$ given by Proposition 1.3.3 that satisfies

- $g(x) = Cx$ if $x \in \partial\Omega$,
- $Dg(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \cap B\left([A,B], \frac{\delta}{2}\right)$ a.e. in Ω, and
- $|\{x \in \Omega : Dg(x) = A\}| > (1-\delta)\lambda|\Omega|$ and $|\{x \in \Omega : Dg(x) = B\}| > (1-\delta)(1-\lambda)|\Omega|$.

Now, since we need a function whose gradient belongs to a neighborhood of $\{A, B\}$ instead of a neighborhood of $[A, B]$, we construct a sequence of functions, $\{g_i\}_{i\in\mathbb{N}}$, such that if

$$U_i = \left\{x \in \Omega : \operatorname{dist}\big(Dg_i, \{A,B\}\big) < (1-2^{-i})\delta\right\}$$

the function g_i is piecewise affine and satisfies

(1) $g_i(x) = Cx$ if $x \in \partial\Omega$,

(2) $Dg_i(x) \in \mathbb{R}^{n\times n}_{\text{sym}} \cap B\left([A,B],(1-2^{-i})\delta\right)$ a.e. in Ω,

(3) $|\{x \in \Omega : Dg(x) = A\}| > (1-\delta)\lambda|\Omega|$ and $|\{x \in \Omega : Dg(x) = B\}| > (1-\delta)(1-\lambda)|\Omega|$,

(4) $g_j(x) = g_i(x)$ for $x \in U_i$ for $j \geqslant i$, and

(5) $|\Omega \setminus U_{i+1}| \leqslant \frac{1}{4}|\Omega \setminus U_i|$ and $U_i \subset U_{i+1}$.

Define g_1 as g, and suppose that g_i has been defined. Using that g_i is piecewise affine, let $\{\tilde{U}_j\}_{j\in\mathbb{N}}$ be open sets and $N \subset \mathbb{R}^n$ such that $|N| = 0$,

$$\Omega \setminus U_i = \left(\bigcup_{j\in\mathbb{N}} \tilde{U}_j\right) \cup N$$

and $Dg_i = \tilde{C}_j$ in $\tilde{U}_j$, for some $\tilde{C}_j \in \mathbb{R}^{n\times n}_{\text{sym}} \cap B([A,B],(1-2^{-i})\delta)$. Hence we can write

$$\tilde{C} = \tilde{\lambda}A + (1-\tilde{\lambda})B + \tilde{D},$$

with $\tilde{\lambda} \in (0,1)$ and $|\tilde{D}| \leqslant (1-2^{-i})\delta$. Set $\tilde{A} = A + \tilde{D}$ and $\tilde{B} = B + \tilde{D}$; then

$$\tilde{C} = \tilde{\lambda}\tilde{A} + (1-\tilde{\lambda})\tilde{B},$$

with $|\tilde{A} - A|, |\tilde{B} - B| \leqslant (1-2^{-i})\delta$ and $\operatorname{rank}(\tilde{A} - \tilde{B}) = \operatorname{rank}(A - B) = 1$. Hence, using Proposition 1.3.3 in each $\tilde{U}_j$ we obtain a piecewise affine function $\tilde{g}_j$ such that

(6) $\tilde{g}_j(x) = \tilde{C}_j x$ if $x \in \partial\tilde{U}_j$,

(7) $D\tilde{g}_j(x) \in \mathbb{R}^{n\times n}_{\text{sym}} \cap B\left([\tilde{A},\tilde{B}],2^{-i-1}\delta\right)$ a.e. in $\tilde{U}_j$,

(8) $\left|\left\{x \in \Omega : D\tilde{g}_j(x) \in \left\{\tilde{A},\tilde{B}\right\}\right\}\right| > \frac{3}{4}|\tilde{U}_j|$.

Define g_{i+1} as g_i in $\Omega \setminus \bigcup_{j\in\mathbb{N}} \tilde{U}_j$ and as $\tilde{g}_j$ in $\tilde{U}_j$. Then, g_{i+1} satisfies the properties (1)-(5) and is piecewise affine. The sequence g_i converges strongly in $W^{1,\infty}$ to a piecewise affine function g_λ that satisfies

(9) $g_\lambda(x) = Cx$ if $x \in \partial\Omega$,

(10) $Dg_\lambda(x) \in \mathbb{R}^{n\times n}_{\text{sym}} \cap B(\{A,B\},\delta)$ a.e. in Ω,

(11) $|\{x \in \Omega : Dg_\lambda(x) = A\}| > (1-\delta)\lambda|\Omega|$ and $|\{x \in \Omega : Dg_\lambda(x) = B\}| > (1-\delta)(1-\lambda)|\Omega|$.

Next, we will manipulate the map g_λ to obtain the exact volume fractions of *(iii)* and *(iv)*. Let

$$\mu_\lambda = \frac{|\{x \in \Omega : |Dg_\lambda(x) - A| < \delta\}|}{|\Omega|};$$

then, $\lambda(1-\delta) < \mu_\lambda < \lambda$. Choose

$$\hat{A} \in [A,B] \cap B\left(A, \frac{\delta}{2}\right)$$

satisfying

$$C = \hat{\lambda}\hat{A} + (1-\hat{\lambda})B,$$

where $\hat{\lambda} = \lambda + \rho$ for some $0 < \rho < 1 - \lambda$. For every $0 < \varepsilon < \frac{\delta}{2}$ we can repeat the above construction changing δ by ε and λ by $\hat{\lambda}$ to obtain a piecewise affine function $g_{\hat{\lambda}}$ that is equal to Cx for $x \in \partial\Omega$, $Dg_{\hat{\lambda}}(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \cap B(\{A,B\},\delta)$ a.e. in Ω and

$$\mu_{\hat{\lambda}} > \hat{\lambda}(1-\varepsilon).$$

Pick $0 < \varepsilon < \frac{\delta}{2}$ such that $\mu_{\hat{\lambda}} > \lambda$, and define

$$t := \frac{\mu_{\hat{\lambda}} - \lambda}{\mu_{\hat{\lambda}} - \mu_{\lambda}}. \tag{1.3.3}$$

Then, $0 < t < 1$ and $\lambda = t\mu_\lambda + (1-t)\mu_{\hat{\lambda}}$. Next, we divide Ω, up to measure zero, into two disjoint domains Ω_λ and $\Omega_{\hat{\lambda}}$ contained in Ω that satisfy $|\Omega_\lambda| = t|\Omega|$ and $|\Omega_{\hat{\lambda}}| = (1-t)|\Omega|$, and cover them with rescaled copies of Ω, i.e., $\Omega_\lambda = \bigcup_{i\in\mathbb{N}} \Omega_{i,\lambda} \cup N_\lambda$ and $\Omega_{\hat{\lambda}} = \bigcup_{i\in\mathbb{N}} \Omega_{i,\hat{\lambda}} \cup N_{\hat{\lambda}}$, where $|N_\lambda| = |N_{\hat{\lambda}}| = 0$ and for all $i \in \mathbb{N}$ there exist $a_{i,\lambda}, a_{i,\hat{\lambda}} \in \mathbb{R}^n$ and $r_{i,\lambda}, r_{i,\hat{\lambda}} > 0$ such that

$$\Omega_{i,\lambda} = a_{i,\lambda} + r_{i,\lambda}\Omega \text{ and } \Omega_{i,\hat{\lambda}} = a_{i,\hat{\lambda}} + r_{i,\hat{\lambda}}\Omega.$$

Denote by $g_{i,\lambda}$ the rescaled copy of g_λ in $\Omega_{i,\lambda}$ and by $g_{i,\hat{\lambda}}$ the rescaled copy of $g_{\hat{\lambda}}$ in $\Omega_{i,\hat{\lambda}}$. We define f by $g_{i,\lambda}$ in $\Omega_{i,\lambda}$ and by $g_{i,\hat{\lambda}}$ in $\Omega_{i,\hat{\lambda}}$. Then, f clearly satisfies *(i)* and $Df(x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \cap B(\{A,B\},\delta)$. Using (1.3.3) we obtain

$$\begin{aligned}
&\left|\left\{x\in\Omega : |Df(x) - A| < \delta\right\}\right| = \sum_{i\in\mathbb{N}} \left(\left|\left\{x\in\Omega : |Dg_{i,\lambda}(x) - A| < \delta\right\}\right| + \left|\left\{x\in\Omega : |Dg_{i,\hat{\lambda}}(x) - A| < \delta\right\}\right| \right) \\
&= \mu_\lambda \sum_{i\in\mathbb{N}} |\Omega_{i,\lambda}| + \mu_{\hat{\lambda}} \sum_{i\in\mathbb{N}} |\Omega_{i,\hat{\lambda}}| = \mu_\lambda|\Omega_\lambda| + \mu_{\hat{\lambda}}|\Omega_{\hat{\lambda}}| = \left(t\mu_\lambda + (1-t)\mu_{\hat{\lambda}}\right)|\Omega| = \lambda|\Omega|.
\end{aligned}$$

Therefore f satisfies *(iii)* and *(iv)*. To obtain *(ii)* we cover Ω with small copies of Ω and we rescale f; as f is Lipschitz, the Hölder norm of $f(x) - Cx$ decreases when we make the copies of Ω smaller. □

In the following proposition we extend the last result to laminates of finite order. Proposition 1.3.5 is at the essence of convex integration: the construction of a function f whose gradient Df is close to a given laminate; moreover, if the laminate is supported in the set of symmetric matrices then Df can also be constructed to be symmetric. In addition, if the laminate is supported in the set of positive definite matrices, then f is a homeomorphism.

Proposition 1.3.5. *Let $n, m, N \in \mathbb{N}\setminus\{0\}$, $A_1,\dots,A_N \in \mathbb{R}^{n\times m}$ and $L \geqslant 1$ be such that*

$$|A_i| \leqslant L, \qquad i = 1,\dots,N.$$

Consider $\lambda_1,\dots,\lambda_N \geqslant 0$ such that $\nu := \sum_{i=1}^N \lambda_i\delta_{A_i}$ is in $\mathscr{L}(\mathbb{R}^{n\times m})$ and call $A := \overline{\nu}$. Then, for every $\alpha \in (0,1)$, $0 < \delta < \frac{1}{2}\min_{1\leqslant i<j\leqslant N}|A_i - A_j|$ and every bounded open set $\Omega \subset \mathbb{R}^n$, there exists a piecewise affine Lipschitz map $f : \Omega \to A\Omega$ such that

(a) $f(x) = Ax$ *for* $x \in \partial\Omega$,

(b) $\|f(x) - Ax\|_{C^\alpha(\overline{\Omega})} < \delta$,

(c) $|\{x \in \Omega : |Df(x) - A_i| < \delta\}| = \lambda_i |\Omega|$ *for all* $i = 1, \dots, N$.

If we also have

$$m = n \text{ and } A_1, \dots, A_N \in \mathbb{R}^{n\times n}_{\mathrm{sym}}, \tag{1.3.4}$$

we can obtain

(d) $Df \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *a.e. in* Ω.

Moreover, if $A_1, \dots, A_N \in \Gamma_+$ *and* L *satisfies*

$$\sigma_1(A_i) \geqslant L^{-1} \text{ for all } i = 1, \dots, N,$$

then, we can choose f *to be a bi-Lipschitz homeomorphism satisfying that*

(e) $\|f^{-1}(x) - A^{-1}x\|_{C^\alpha(A\overline{\Omega})} < \delta$.

for all $\delta < \delta_0$, *where* $\delta_0 < \frac{1}{2}\min_{1\leqslant i<j\leqslant N} |A_i - A_j|$ *may depend on* L.

Proof. Proposition 1.3.4 proves *(a)-(d)* for $N = 2$. For a general $N \in \mathbb{N}$ we proceed by induction. Suppose that we have proved *(a)-(d)* for N and we will prove them for $N+1$. We will use convex integration in the same spirit as in [126, Lemma 3.2] but working with the C^α norm instead of the C^0 norm.

Given a laminate of finite order $\nu = \sum_{i=1}^{N+1} \lambda_i \delta_{A_i}$, let $\nu' = \sum_{i=1}^{N} \lambda_i' \delta_{A_i'}$ be a laminate of finite order such that ν is obtained from ν' by elementary splitting, i.e., there exists $\lambda \in (0,1)$ such that, for $i \in \{1, \dots, N-1\}$ we have $A_i = A_i'$ and $\lambda_i = \lambda_i'$, $\mathrm{rank}(A_N - A_{N+1}) = 1$, $A_N' = \lambda A_N + (1-\lambda)A_{N+1}$, $\lambda_N = \lambda_N'\lambda$ and $\lambda_{N+1} = \lambda_N'(1-\lambda)$. Then if we write $B := A_N$ and $C := A_{N+1}$ we have

$$\nu = \sum_{i=1}^{N-1} \lambda_i' \delta_{A_i'} + \lambda_N'\lambda\delta_B + \lambda_N'(1-\lambda)\delta_C.$$

Since ν' has order $N-1$, by induction assumption, there exists a piecewise affine function g satisfying *(a)-(c)* for $\frac{\delta}{2}$ and, if (1.3.4) holds, it also satisfies *(d)*. Let

$$\omega = \left\{ x \in \Omega : |Dg - A_N'| < \frac{\delta}{2} \right\}.$$

Since g is piecewise affine we have that there exist $\{M_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^{n\times m}$ ($\mathbb{R}^{n\times n}_{\mathrm{sym}}$ if (1.3.4) is satisfied) and $\{\omega_j\}_{j\in\mathbb{N}}$ disjoint open sets contained in ω, such that $|\omega \setminus \bigcup_{j\in\mathbb{N}} \omega_j| = 0$ and $Dg(x) = M_j$ for $x \in \omega_j$. Hence $|M_j - A_N'| < \frac{\delta}{2}$ for all $j \in \mathbb{N}$. For $j \in \mathbb{N}$. Let

$$B_j = B - A_N' + M_j \text{ and } C_j = C - A_N' + M_j.$$

Then $M_j = \lambda B_j + (1-\lambda)C_j$ and thanks to Proposition 1.3.4, there is a piecewise affine function h_j such that $h_j(x) = M_j x$ for $x \in M_j$, $\|h_j(x) - M_j x\|_{C^\alpha(\overline{\omega}_j)} < \frac{\delta}{2}$,

$$\left|\left\{x \in \omega_j : |Dh_j(x) - B_j| < \frac{\delta}{2}\right\}\right| = \lambda|\omega_j|,$$

$$\left|\left\{x \in \omega_j : |Dh_j(x) - C_j| < \frac{\delta}{2}\right\}\right| = (1-\lambda)|\omega_j|,$$

and, in case (1.3.4) is satisfied, we also have $Dh_j \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Then, define f as

$$f(x) := \begin{cases} g(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_{j\in\mathbb{N}} \omega_j, \\ h_j(x) & \text{if } x \in \omega_j \text{ for some } j \in \mathbb{N}. \end{cases}$$

Using $0 < \delta < \frac{1}{2}\min_{1\leqslant i<j\leqslant N} |A_i - A_j|$, $|B_j - B| < \frac{\delta}{2}$, $|\omega| = \lambda'_N$ and that g satisfies

$$\left|\left\{x \in \Omega : |Dg(x) - A'_i| < \frac{\delta}{2}\right\}\right| = \lambda'_i|\Omega| \quad \text{for all } i = 1,\dots,N,$$

we get

$$\left|\left\{x \in \Omega : |Df(x) - B| < \delta\right\}\right| = \sum_{j\in\mathbb{N}} \left|\left\{x \in \omega_j : |Dh_j(x) - B_j| < \frac{\delta}{2}\right\}\right| = \lambda \sum_{j\in\mathbb{N}} |\omega_j| = \lambda\lambda_N|\Omega|.$$

In the same way, using $|C_j - C| < \frac{\delta}{2}$ instead of $|B_j - B| < \frac{\delta}{2}$ we obtain

$$\left|\left\{x \in \Omega : |Df(x) - C| < \delta\right\}\right| = (1-\lambda)\lambda_N|\Omega|.$$

Therefore

$$|\{x \in \Omega : |Df(x) - A_i| < \delta\}| = \lambda_i|\Omega| \quad \text{for all } i = 1,\dots,N+1$$

and we have proved *(c)*. Property *(a)* is immediate from the fact that $g(x) = Ax$ for $x \in \Omega$ and the definition of f. The Hölder continuity, that is, property *(b)*, comes from Lemma 1.2.3. And, if (1.3.4) is satisfied we use that, in such a case, we have $Dg \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ a.e. in Ω and $Dh_j \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ in ω_j, to get *(d)*.

Now, we will suppose that we have $A_1,\dots,A_N \in \Gamma_+$ and that L satisfies

$$\sigma_1(A_i) \geqslant L^{-1} \text{ for all } i = 1,\dots,N,$$

and we will prove that the f just constructed is in fact a bi-Lipschitz homeomorphism that satisfies *(e)*.

To prove this, we first show that f is bi-Lipschitz. We extend f to an open ball Ω' such that $\overline{\Omega} \subset \Omega'$ and $f(x) = Ax$ in $\Omega' \setminus \Omega$. Thus, f is continuous in Ω'. We define, for each $\varepsilon > 0$,

$$\Omega'_\varepsilon = \{x \in \Omega' : \mathrm{dist}(x, \partial\Omega') > \varepsilon\}.$$

By *(c)* we get that $Df(x) \geqslant \frac{1}{2L} I$, and $|Df(x)| \leqslant 2L$ a.e. in Ω'. As Ω' is convex then f is $2L$-Lipschitz. Let $\{\eta_\varepsilon\}_{0<\varepsilon\leqslant 1}$ be a standard family of mollifiers, and $f_\varepsilon := \eta_\varepsilon * f \in C^\infty(\Omega'_\varepsilon)$ the mollification of f.

Using that the matrices $M \in \mathbb{R}^{n\times n}_{\text{sym}}$ satisfying $(2L)^{-1} I \leqslant M$ form a convex set, we find that there exists an $\varepsilon_0 > 0$ such that if $\varepsilon \leqslant \varepsilon_0$ then $Df_\varepsilon(x) \geqslant \frac{1}{2L} I$ in $\Omega'_\varepsilon \supset \overline{\Omega}$.

For each $x, y \in \Omega'_\varepsilon$, calling $h = y - x$, we have that

$$\begin{aligned}|f_\varepsilon(y) - f_\varepsilon(x)||h| \geqslant |\langle f_\varepsilon(y) - f_\varepsilon(x), h\rangle| &= \left|\left\langle \int_0^1 Df_\varepsilon(x+th)h\,dt, h\right\rangle\right| \\ &\geqslant \int_0^1 \langle Df_\varepsilon(x+th)h, h\rangle\,dt \geqslant \int_0^1 \frac{1}{2L}|h|^2\,dt = \frac{1}{2L}|h|^2,\end{aligned}$$

that is,

$$|f_\varepsilon(y) - f_\varepsilon(x)| \geqslant \frac{1}{2L}|x-y|.$$

Using that $f_\varepsilon \to f$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$, we get

$$\frac{1}{2L}|x-y| \leqslant |f(y) - f(x)|.$$

Hence f is K-bi-Lipschitz in $\overline{\Omega}$ with $K = 2L$. Therefore, f is a homeomorphism onto its image. The equalities $f(\Omega) = A\Omega$ and $f(\overline{\Omega}) = A\overline{\Omega}$ follows from standard results using the topological degree (e.g., [11, Theorems 1 and 2]).

Now we will estimate the C^α seminorm of $f^{-1} - A^{-1}$. For each $y_1, y_2 \in f(\overline{\Omega})$, let $x_i = f^{-1}(y_i)$, $i = 1,2$. Using that f is K-bilipschitz and that $A \geqslant L^{-1} I$, we get

$$\begin{aligned}\frac{|f^{-1}(y_1) - A^{-1}y_1 - f^{-1}(y_2) + A^{-1}y_2|}{|y_1 - y_2|^\alpha} &\leqslant K^\alpha |A^{-1}| \frac{|Ax_1 - f(x_1) - Ax_2 + f(x_2)|}{|x_1 - x_2|^\alpha} \\ &\leqslant K^\alpha |A^{-1}| \left|f - A\right|_{C^\alpha(\overline{\Omega})} \leqslant K^\alpha L\delta = 2L^{\alpha+1}\delta,\end{aligned}$$

so

$$|f^{-1} - A^{-1}|_{C^\alpha(A\overline{\Omega})} \leqslant 2\delta L^{\alpha+1}.$$

Therefore, $|f^{-1} - A^{-1}|_{C^\alpha(A\overline{\Omega})}$ can be done as small as we wish. Finally,

$$\|f^{-1} - A^{-1}\|_{L^\infty(A\overline{\Omega})} \leqslant |f^{-1} - A^{-1}|_{C^\alpha(A\overline{\Omega})} (\operatorname{diam} A\Omega)^\alpha,$$

so $\|f^{-1} - A^{-1}\|_{C^\alpha(A\overline{\Omega})}$ is as small as we wish. □

Chapter 2

Sobolev homeomorphisms with gradients of low rank and sharp integrability

In this chapter we will deal with the Luzin condition. A capital role is played by it and its lower dimensional counterparts. Given $\Omega \subset \mathbb{R}^n$, we say that $f : \Omega \to \mathbb{R}^n$ satisfies the m-dimensional Luzin's condition if for almost every $y \in \mathbb{R}^{n-m}$ and every $A \subset \Omega \cap (\mathbb{R}^m \times \{y\})$ with $\mathscr{H}^m(A) = 0$, we have $\mathscr{H}^m(f(A)) = 0$. The critical exponent to satisfy this condition is $p = m$, and, as in the case of the classical Luzin's condition (i.e., the n-dimensional), there are many sufficient hypotheses that, combined with $f \in W^{1,m}(\Omega)$, give the m-dimensional Luzin's condition. In particular, if $f \in W^{1,m}(\Omega)$ is Hölder continuous, then it satisfies the m-dimensional Luzin's condition. Another typical hypothesis for the n-dimensional Luzin's condition is that f is a homeomorphism; this also holds for $m = 1$ and $m = n-1$ [38], but not for $2 \leqslant m \leqslant n-1$ as is shown in [83].

Sufficient conditions to satisfy the Luzin's condition and counterexamples to it have been largely studied. We would like to mention [82] where Hencl proves that there exists a homeomorphism f in $W^{1,p}((0,1)^n, (0,1)^n)$, $1 \leqslant p < n$, whose Jacobian determinant Jf equals zero a.e. Notice that condition $p < n$ is necessary, since if $f \in W^{1,n}$ and $Jf \geqslant 0$, then f satisfies the Luzin (N) condition. Consequently, the area formula holds. Therefore, any $f \in W^{1,n}$ with $Jf = 0$ a.e. satisfies

$$|f(\Omega)| \leqslant \int_\Omega Jf = 0,$$

and, hence, f cannot be a homeomorphism.

An interesting positive result can be found in [97] where it is proved that if a Sobolev map f is such that

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |Df|^{n-\varepsilon} = 0, \tag{2.0.1}$$

then f satisfies Luzin's condition. The construction of [24] elaborates on that of [82] to show that there exists a homeomorphism $f \in W^{1,1}((0,1)^n, \mathbb{R}^n)$ such that $Jf = 0$ almost everywhere and Df is in the grand Lebesgue space $L^{n)}$, i.e.,

$$\sup_{0<\varepsilon\leq n-1} \varepsilon \int_{(0,1)^n} |Df|^{n-\varepsilon} < \infty.$$

Obviously, such f cannot satisfy (2.0.1); otherwise, f will satisfy the Luzin condition, and therefore $|f(\Omega)| = 0$, which contradicts the fact that f is a homeomorphism.

Given $f \in W^{1,n}(\Omega)$, other sufficient hypotheses to satisfy the classical Luzin's condition can be found in [109]. In [101] we can find conditions regarding the modulus of continuity of a function $f \in W^{1,m}(\Omega)$ to satisfy the m-dimensional Luzin condition. The counterexamples of the m-dimensional Luzin condition are related to functions with derivatives of low rank. Examples of these functions can be found in [25, 56, 106, 129].

Another interesting question is what happens with the inverse of these maps. In [50] it was constructed a bi-Sobolev function with the function and the inverse having zero Jacobian a.e. Černý [25] constructed a bi-Sobolev homeomorphism f with zero minors of Df and Df^{-1} almost everywhere.

All those constructions of Hencl and Černý were based on a careful explicit construction and a limit process to obtain a Cantor set where the distributional Jacobian is supported.

The violation of the Luzin condition is related with the problem of when Det = det and all the counterexamples to the Luzin's condition mentioned in this introduction also are counterexamples to Det = det. In fact, the m-dimensional Luzin condition is related to the equality between the m-dimensional distributional minors and the pointwise minors.

In this chapter we explore a different way of obtaining such kind of pathological maps by using the staircase laminates invented by Faraco [54], in combination with the tools developed in Chapter 1. In fact, such laminates have turned out to be useful in a number of apparently unrelated problems such as L^p theory of elliptic equations [6], Burkholder functions [20], Hessians of rank-one convex functions [35], microstructure and phase transitions in solids [131, 121] and counterexamples of L^1 estimates [34]. As in the case of Ornstein inequalities [34], laminates allow one to decouple the construction of pathological maps into an analytical part and a geometrical part. The analytical part is taken care by the general theory of laminates and the version of convex integration, see Chapter 1. The geometrical part, which is the key in the whole process, consists in finding a suitable staircase laminate.

The structure of the chapter is as follows. In Section 2.1 we present some positive results that relate the integrability and the rank of the derivative of a homeomorphism f; we also relate the integrability of Df with the rank of Df^{-1}. In Section 2.2 we use a small variant of the laminates appeared in [6] with the difference that we want our laminates supported in the coordinate axis instead of being supported in a cone. In fact, in [55] it was sketched how to use laminates to obtain, in dimension 2, a convex function whose gradient f is in $W^{1,p}$ for all $p < 2$, and satisfies $Jf = 0$ a.e., recovering another interesting example of Alberti and Ambrosio [2]. In Section 2.3 we construct a bi-Sobolev homeomorphism f with Df and Df^{-1} of low rank, such that Df and Df^{-1} also have the highest integrability possible according to the results of Section 2.1. As this homeomorphism will be bi-Sobolev, we need to have a control over the inverse; to do so, we use other laminates different of the laminates used in Section 2.2. In these last two sections we will use the tools developed in Chapter 1. The results of this chapter are based in [56] and [129].

We want to mention the work of Liu and Malý, [106], where they prove a result very similar to Theorem 2.2.1, with a construction related to laminates but not inspired by the techniques of Chapter 1. They construct a strictly convex function $u : (0,1)^n \to \mathbb{R}$ such that $u \in W^{2,p}$ for every $p < k$ and $\operatorname{rank} D^2 u < k$. This work appeared at the same time as [56], where Theorem 2.2.1 is proved.

2.1 Sharp integrability for maps with low rank derivative

In this section we will give some results about is the highest integrability of the derivative of a homeomorphism with the derivative being of low rank. This will prove that the integrability of the derivative of the homeomorphisms constructed in Section 2.2 is sharp.

In the following result we focus ourselves on the case when f is Sobolev but we do not know anything about f^{-1}. Theorem 2.1.1 can be found in [56]. In the particular cases when m is equal to 1, $n-1$ or n we can remove from condition a) the Hölder continuity. Cases $m=1$ and $m=n$ are classic, and case $m=n-1$ can be found in [38]. This cannot be done for different values of m as it is shown in [83].

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $m \in \{2,\dots,n\}$. Let $f:\overline{\Omega}\to\mathbb{R}^n$ be a homeomorphism in $W^{1,p}(\Omega,\mathbb{R}^n)$ such that $f|_{\partial\Omega} = \mathrm{id}|_{\partial\Omega}$. Assume one of the following:*

a) $p=m$ and f is Hölder continuous.

b) $p>m$.

Then $\mathrm{rank}(Df(x)) \geqslant m$ *for all x in a subset of Ω of positive measure.*

To prove this theorem we need to show the validity of the area formula for restrictions to planes of dimension m. Given $f \in W^{1,p}(\Omega,\mathbb{R}^n)$, we define its k-dimensional Jacobian $J_k f(x)$ as $\left(\sum_M |M|^2\right)^{\frac{1}{2}}$, where the sum runs over all the minors of $Df(x)$ of order k. We denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure in $\mathbb{R}^n$; when $\mathcal{H}^m$ acts on subsets of coordinate planes of dimension m, it can be identified with the Lebesgue measure in $\mathbb{R}^m$.

Lemma 2.1.2. *Let $f:\overline{\Omega}\to\mathbb{R}^n$ be injective and in $W^{1,p}(\Omega,\mathbb{R}^n)$. Assume that one of the alternatives a)–b) of Theorem 2.1.1 holds. Then for almost every $y\in\mathbb{R}^{n-m}$ and all $\mathcal{H}^m$-measurable sets $E\subset\Omega\cap\left(\mathbb{R}^m\times\{y\}\right)$,*

$$\int_E J_m f\, d\mathcal{H}^m = \mathcal{H}^m\left(f(E)\right).$$

Proof. By Fubini's theorem, for a.e. $y\in\mathbb{R}^{n-m}$, the restriction $f|_{\Omega\cap(\mathbb{R}^m\times\{y\})}$ is in $W^{1,p}$ with respect to the $\mathcal{H}^m$ measure. Fix such a y. A standard approximation theorem (see, e.g., [53, Corollary 6.6.2]) shows that there exist sequences $\{f_j\}_{j\in\mathbb{N}}$ in $C^1(\mathbb{R}^n,\mathbb{R}^n)$ and $\{E_j\}_{j\in\mathbb{N}}$ of disjoint $\mathcal{H}^m$-measurable subsets of $E\cap\left(\mathbb{R}^m\times\{y\}\right)$ such that

$$f(x)=f_j(x) \quad\text{and}\quad Df(x)=Df_j(x) \qquad \text{for all } x\in E_j \text{ and } j\in\mathbb{N}$$

and $\mathcal{H}^m(E\setminus\bigcup_{j=1}^\infty E_j)=0$. Thus,

$$\int_E J_m f\, d\mathcal{H}^m = \sum_{j=1}^\infty \int_{E_j} J_m f\, d\mathcal{H}^m.$$

Now, for each $j\in\mathbb{N}$, thanks to the area formula for regular maps (see [59, Theorem 3.2.3]) and the fact that f is injective,

$$\begin{aligned}\int_{E_j} J_m f\, d\mathcal{H}^m &= \int_{E_j} J_m f_j\, d\mathcal{H}^m = \int_{\mathbb{R}^n} \mathrm{Card}\{E_j\cap f_j^{-1}(y)\}\, d\mathcal{H}^m(y)\\ &= \int_{\mathbb{R}^n} \mathrm{Card}\{E_j\cap f^{-1}(y)\}\, d\mathcal{H}^m(y) = \mathcal{H}^m(f(E_j)).\end{aligned}$$

Therefore,

$$\int_E J_m f \, d\mathcal{H}^m = \mathcal{H}^m\Big(f\big(\bigcup_{j=1}^{\infty} E_j\big)\Big).$$

Under assumptions *a)* or *b)* of Theorem 2.1.1, $f|_{\Omega\cap(\mathbb{R}^m\times\{y\})}$ satisfies the m-dimensional Luzin (N) condition, i.e., given $A \subset \Omega \cap \left(\mathbb{R}^m \times \{y\}\right)$ such that $\mathcal{H}^m(A) = 0$ then $\mathcal{H}^m\left(f(A)\right) = 0$. The proof under *a)* is due to [101, Theorem 1.1] (with $\lambda = 0$ in the notation there), while the proof under *b)* is classical [113]. In either case,

$$\mathcal{H}^m\Big(f\big(E \setminus \bigcup_{j=1}^{\infty} E_j\big)\Big) = 0$$

and the proof is concluded. □

Now we can prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Suppose, for a contradiction, that $\operatorname{rank}(Df(x)) < m$ for a.e. $x \in \Omega$. Then $J_m f = 0$ a.e., thanks to the Cauchy–Binet formula. Then, for a.e. $y \in \mathbb{R}^{n-m}$ we have $J_m f = 0$ $\mathcal{H}^m$-a.e. in $\Omega \cap \left(\mathbb{R}^m \times \{y\}\right)$ and, by Lemma 2.1.2,

$$\mathcal{H}^m\left(f\left(\Omega \cap \left(\mathbb{R}^m \times \{y\}\right)\right)\right) = 0. \tag{2.1.1}$$

Let $P_m : \mathbb{R}^n \to \mathbb{R}^m$ be the projection over the first m coordinates: $P_m(x_1,\dots,x_n) = (x_1,\dots,x_m)$, and, for any $y \in \mathbb{R}^{n-m}$, define the set $\Omega_y = P_m\left(\Omega \cap \left(\mathbb{R}^m \times \{y\}\right)\right)$ and the function $g_y : \Omega_y \to \mathbb{R}^m$

$$g_y(x) = P_m\left(f\left(x, y\right)\right).$$

Since $f = \mathrm{id}$ on $\partial\Omega$ and $\partial\Omega_y = P_m\left(\partial\Omega \cap \left(\mathbb{R}^m \times \{y\}\right)\right)$, we have that $g_y = \mathrm{id}$ on $\partial\Omega_y$. Using now degree theory, this implies

$$\deg\left(g_y, \Omega_y, \cdot\right) = \deg\left(\mathrm{id}, \Omega_y, \cdot\right)$$

and, consequently, $\Omega_y \subset g_y(\Omega_y)$ (see, e.g., [45, Theorem 3.1]). Fix $y \in \mathbb{R}^{n-m}$ such that (2.1.1) holds and $\Omega_y \neq \emptyset$. As P_m is 1-Lipschitz and Ω_y is open, we find that

$$0 < \mathcal{H}^m\left(\Omega_y\right) \leqslant \mathcal{H}^m\left(g_y\left(\Omega_y\right)\right) \leqslant \mathcal{H}^m\left(f\left(\Omega \cap \left(\mathbb{R}^m \times \{y\}\right)\right)\right),$$

which contradicts (2.1.1) and completes the proof. □

As can be seen from the proof, in Theorem 2.1.1 and Lemma 2.1.2, conditions *a)*–*b)* can be replaced by any other assumption implying m-dimensional Luzin's condition. As mentioned in the proof, the paper [101] shows some of those conditions.

For bi-Sobolev functions we can show the following results, which can be found in [129].

Theorem 2.1.3. *Let $f : \Omega \to \mathbb{R}^n$ be a bi-Sobolev homeomorphism such that $f^{-1} \in W^{1,q}\left(f(\Omega), \mathbb{R}^n\right)$ and for a measurable set $E \subset \Omega$ we have $Jf = 0$ almost everywhere on E. Let $m \in \{1,\dots,n-1\}$ and assume one of the following:*

a) *$q = m$ and f^{-1} is Hölder continuous.*

b) $q > m$.

c) $q = m = n-1$.

Then $\operatorname{rank}(Df) \leqslant n-m-1$ *almost everywhere on* E.

Observe that this implies that given a bi-Sobolev homeomorphism $f \in W^{1,p}$ with $p \geqslant n-1$, we have that $Jf^{-1}(y)$ cannot be 0 for a.e. $y \in f(\Omega)$.

Using the last theorem and Theorem 2.1.1 we obtain the following two results. The first one gives an upper bound for the sum of the integrabilities of Df and Df^{-1} for a bi-Sobolev homeomorphism f with Jacobian equal to zero almost everywhere.

Theorem 2.1.4. *Let* $f : \Omega \to \mathbb{R}^n$ *be a bi-Sobolev homeomorphism such that* $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $f^{-1} \in W^{1,q}\big(f(\Omega), \mathbb{R}^n\big)$, $f|_{\partial\Omega} = \mathrm{id}|_{\partial\Omega}$ *and* $Jf = 0$ *almost everywhere in* Ω. *Then* $p + q \leqslant n+1$. *Moreover, if* $p + q = n+1$ *then* $p, q \in \mathbb{N}$ *and* $f, f^{-1} \notin C^\alpha$ *for any* $\alpha \in (0,1)$.

We do not know whether there exists a bi-Sobolev homeomorphism with $Jf = 0$ a.e. that attains the equality $p + q = n+1$. Moreover, we do not know if there exists a bi-Sobolev homeomorphism f satisfying $Jf = 0$ a.e., and a natural number $m \in \{2, \dots, n-2\}$ such that $f \in W^{1,m}$ and $f^{-1} \in W^{1,n-m}$.

Finally, the next theorem shows that Theorem 2.3.1 (in Section 2.3 below), where we construct a bi-Sobolev homeomorphism f with Df and Df^{-1} of low rank, is sharp and gives bounds for the integrability of Df and Df^{-1} depending on the rank of Df and Df^{-1}.

Theorem 2.1.5. *Let* $m_1, m_2 \in \mathbb{N}$ *be such that* $m_1, m_2 \leqslant n-1$, *let* $f : \Omega \to \mathbb{R}^n$ *be a bi-Sobolev homeomorphism such that* $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $f^{-1} \in W^{1,q}\big(f(\Omega), \mathbb{R}^n\big)$, $f|_{\partial\Omega} = \mathrm{id}|_{\partial\Omega}$, $\operatorname{rank}(Df) \leqslant m_1$, $\operatorname{rank}(Df^{-1}) \leqslant m_2$ *almost everywhere in* Ω *and suppose that there exist measurable* $A, B \subset \Omega$ *with* $|A|, |B| > 0$, $\operatorname{rank}(Df) = m_1$ *on* A *and* $\operatorname{rank}(Df^{-1}) = m_2$ *on* B. *Then* $p \leqslant \min\{m_1 + 1, n - m_2\}$ *and* $q \leqslant \min\{m_2 + 1, n - m_1\}$.

Observe that this theorem implies that $p + q \leqslant n+1$.

The proof of Theorem 2.1.3 follows that in [91, Theorem 4].

Proof of Theorem 2.1.3. Suppose that $\operatorname{rank}(Df) \geqslant n-m$ in a set $A \subset E$ of positive measure. Then $|J_{n-m} f| > 0$ on A, and without loss of generality we can assume that $Jf = 0$ on A and that f is Lipschitz on A, see [53, Section 6.6, Theorem 3].

For each $I \subset \{1, \dots, n\}$ with $|I| = n-m$ let $\pi_I : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be the projection that sends $x = (x_1, \dots, x_n)$ to $(x_{i_1}, \dots, x_{i_{n-m}})$, where $I = \{i_1, \dots, i_{n-m}\}$ and $i_1 < \dots < i_{n-m}$. Define $h_I = \pi_I \circ f$ and set $P(z) = \pi_I^{-1}(z) \cap f(\Omega)$ for $z \in \mathbb{R}^{n-m}$. Since $|J_{n-m} f| > 0$ in A, there exists $I \subset \{1, \dots, n\}$ such that $|I| = n-m$ and $|J_{n-m} h_I| > 0$ on a subset of A of positive measure, still called A. Since f is Lipschitz on A, we can use the area formula (see, e.g. [53], [59], [3]) to conclude that

$$|f(A)| = 0.$$

Using the coarea formula we get

$$0 < \int_A |J_{n-m} h_I| = \int_{\mathbb{R}^{n-m}} \mathcal{H}^m\left(\{x \in A : h_I(x) = z\}\right) dz.$$

Hence, for a set $F \subset \mathbb{R}^{n-m}$ of positive measure we have that $\forall z \in F$,

$$\mathcal{H}^m \left(f^{-1}\left(f(A) \cap P(z)\right)\right) = \mathcal{H}^m \left(\{x \in A : h_I(x) = z\}\right) > 0. \tag{2.1.2}$$

In the equality we have used that f is injective. On the other hand, since $|f(A)| = 0$ it follows that for $\mathcal{H}^{n-m}$-almost every $z \in \mathbb{R}^{n-m}$ we get $\mathcal{H}^m\left(f(A) \cap P(z)\right) = 0$.

As $f^{-1} \in W^{1,q}$, we have that $f^{-1} \in W^{1,q}(P(z))$ for $\mathcal{H}^{n-m}$-almost every $z \in \mathbb{R}^{n-m}$, so under any option *a), b)* or *c)*, $f^{-1}|_{P(z)}$ satisfies the m-dimensional Luzin (N) condition. The proof under *a)* is due to [101, Theorem 1.1] (with $\lambda = 0$ in the notation there), the result under *b)* is classical [113], and the proof under *c)* is due to [38, Theorem 1.3]. Therefore, for $\mathcal{H}^{n-m}$-almost every $z \in \mathbb{R}^{n-m}$, we obtain

$$\mathcal{H}^m \left(f^{-1}\left(f(A) \cap P(z)\right)\right) = 0,$$

and we have a contradiction with (2.1.2). □

Now using Theorem 2.1.3 we are able to prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Define $m_p = \lceil p \rceil - 1$ and $m_q = \lceil q \rceil - 1$. Observe that $m_p < p \leqslant m_p + 1$ and $m_q < q \leqslant m_q + 1$. Then, using Theorem 2.1.3, we have that $\operatorname{rank} Df \leqslant n - m_q - 1$ almost everywhere on Ω. On the other hand, using [56, Theorem 12], we get that $\operatorname{rank} Df \geqslant m_p$ in a subset of Ω of positive measure; we also obtain $\operatorname{rank} Df^{-1} \geqslant m_q$. Therefore, we have

$$m_p \leqslant n - m_q - 1.$$

Hence

$$p + q \leqslant m_p + m_q + 2 \leqslant n + 1. \tag{2.1.3}$$

If $p + q = n + 1$ then $p = m_p + 1$ and $q = m_q + 1$. If, in addition, f^{-1} or f were Hölder continuous then, by Theorem 2.1.3, $\operatorname{rank}(Df) \leqslant n - q - 1$ or $\operatorname{rank}(Df^{-1}) \leqslant n - p - 1$, but this contradicts $\operatorname{rank}(Df) \geqslant m_p = p - 1$ in the first case and $\operatorname{rank}(Df^{-1}) \geqslant m_q = q - 1$ in the second case. □

Proof of Theorem 2.1.5. First, we observe that thanks to [56, Theorem 12] we obtain $p \leqslant m_1 + 1$ and $q \leqslant m_2 + 1$.

On the other hand, denote $m = \lceil q \rceil - 1$, then $m < q \leqslant m + 1$, and using Theorem 2.1.3 with $E = A$ we obtain $\operatorname{rank}(Df) \leqslant n - m - 1 \leqslant n - q$ almost everywhere on A, so $q \leqslant n - m_1$. In the same way, we get $p \leqslant n - m_2$. The theorem follows. □

2.2 Sobolev homeomorphisms with gradients of low rank

In the current section we show that with staircase laminates it is possible to combine the results of [55, 2, 82] and the result of Černý [25], where he constructs a homeomorphism f with all its minors of m-th order equal to zero almost everywhere belonging to $W^{1,p}$ with $1 \leqslant p < \frac{n}{n+1-m}$. In Subsection 2.2.1 we give a sketch of the proof of Theorem 2.2.1 with the intention to make the proof clearer (Subsections 2.2.2 and 2.2.3).

To be precise, we build a probability measure formed by staircase laminates in the planes parallel to the coordinate axes, which can be pushed to show that not only the Jacobian is zero but also that Df has all its minors of order m equal to zero almost everywhere and Df is in $L^{m,w}$, i.e., there exists a constant $C>0$ such that

$$|\{x \in \Omega : |Df(x)| > t\}| \leqslant Ct^{-m}, \qquad t > 0.$$

In the particular case when $m = n$ we have $L^{n,w} \subset L^{n)}$, so Theorem 2.2.1 is sharp in the sense of (2.0.1). For $m \in \{2,\ldots,n-1\}$ the result is sharp in the sense of Theorem 2.1.1.

The main theorem of this section is the following.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $m \in \mathbb{N}$, $2 \leqslant m \leqslant n$, $\delta > 0$ and $\alpha \in (0,1)$. Then there exists a strictly convex function $u : \Omega \to \mathbb{R}$, $u \in W^{2,1}(\Omega)$, whose gradient $f = \nabla u$ satisfies:*

i) $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ *and* $f : \Omega \to \Omega$ *is a homeomorphism.*

ii) $f = \mathrm{id}$ *on* $\partial\Omega$.

iii) $\mathrm{rank}(Df(x)) < m$ *for a.e.* $x \in \Omega$.

iv) $Df \in L^{m,w}(\Omega, \mathbb{R}^{n\times n})$.

v) $\|f - \mathrm{id}\|_{C^\alpha(\overline{\Omega})} < \delta$ *and* $\|f^{-1} - \mathrm{id}\|_{C^\alpha(\overline{\Omega})} < \delta$.

Notice that in the case $n = m$, as explained in [82], using the area formula for Sobolev maps ([72]) we have that this kind of homeomorphisms sends a set of full measure to a null set, and a null set to a set of full measure, i.e., there exists $Z \subset \Omega$ of measure zero such that

$$|f(\Omega \setminus Z)| = \int_{f(\Omega\setminus Z)} dy = \int_{\Omega\setminus Z} Jf(x)\,dx = 0$$

and

$$|f(Z)| = |f(\Omega)| - |f(\Omega \setminus Z)| = |f(\Omega)|.$$

2.2.1 Sketch of the construction of the Sobolev homeomorphism

In this subsection we construct the sequence of laminates ν_k of finite order that is behind the construction of the Sobolev homeomorphism with low rank derivative. The actual proof will consist in the following steps:

- Construct the sequence of laminates ν'_k. While in the sketch of the proof the laminates ν_k are supported in non-invertible matrices, in the real proof the ν'_k are supported in positive definite matrices; nevertheless, the ν'_k approximate the ν_k of the sketch.
- Use Proposition 1.3.5 to obtain homeomorphisms that are close to the laminates ν'_k in small regions of the domain.
- Paste the local homeomorphisms in the small regions to obtain a global homeomorphism f_j in $\bar{\Omega}$.
- Pass to the limit in the homeomorphisms f_j to obtain the homeomorphism f of Theorem 2.2.1. The fact that f is the gradient of a convex function is Lemma 1.3.1.

As mentioned before, this section presents a simplified construction of the laminates, and will help the reader to follow Subsection 2.2.2.

In order to construct ν_k, we need to define the sets

$$S_i^k = \left\{A = k\,\mathrm{diag}(v) : v \in \{0,1\}^n \text{ and } \mathrm{rank}\,(A) = n-i\right\}, \qquad k \in \mathbb{N}, \quad i \in \{0,\dots,n-m\}$$

and

$$E = \{A \in \mathbb{R}^{n\times n} : \mathrm{rank}(A) < m\}.$$

Thus, the matrices of S_i^k are k times the identity matrix in the $(n-i)$-dimensional linear subspaces parallel to the coordinate axes.

The main property of the laminates to be constructed is as follows: for each $k \in \mathbb{N}$, ν_k is supported in $\bigcup_{i=0}^{n-m} S_i^k \cup E$,

$$|A| \leqslant k \text{ for all } A \in \mathrm{supp}\,(\nu_k)\,,$$

and

$$\nu_k\left(S_i^k\right) \leqslant C k^{i-n},$$

for some $C > 0$.

The weak* limit ν of ν_k is supported in the set E. It satisfies that there exists a constant $C > 0$ such that

$$\nu\,(\{|A| > t\}) \leqslant C t^{-m}, \qquad t > 0. \tag{2.2.1}$$

This last inequality will give us the desired integrability of the derivative of the homeomorphism.

The laminates ν_k are defined inductively as follows. We start with $\nu_1 = \delta_I$. Now, given

$$\nu_{k-1} = \sum_{j=1}^{N} \lambda_j \delta_{A_j} \in \mathscr{L}(\mathbb{R}^{n\times n}), \tag{2.2.2}$$

with $A_j \in \bigcup_{i=0}^{n-m} S_i^{k-1} \cup E$, all different, we are going to split the matrices of S_i^{k-1} in matrices in $\bigcup_{i=0}^{n-m} S_i^k \cup E$ following rank-one lines as in Definition 1.0.1.

Let $A \in \mathrm{supp}\,(\nu_{k-1})$. If $A \in E$ we define $\nu_A = \delta_A$, whereas if $A \in S_i^{k-1}$ for some $i \in \{0,\dots,n-m\}$, we construct ν_A inductively. Without loss of generality,

$$A = (k-1)\,\mathrm{diag}(\underbrace{1,\dots,1}_{n-i},\underbrace{0,\dots,0}_{i}).$$

Claim 1. *There exist families*

$$\{B_{\ell,j}\}_{\substack{\ell=0,\dots,n-i\\ j=0,\dots,2^\ell-1}} \subset \mathbb{R}^{n\times n} \quad \textit{and} \quad \{\lambda_{\ell,j}\}_{\substack{\ell=0,\dots,n-i\\ j=0,\dots,2^\ell-1}} \subset [0,1] \tag{2.2.3}$$

such that, for $0 \leqslant \ell \leqslant n-i$ *and* $0 \leqslant j \leqslant 2^\ell - 1$, $B_{\ell,j}$ *are diagonal,* $\lambda_{\ell,j} \geqslant 0$,

$$\sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} = 1, \qquad A = \sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} B_{\ell,j}, \tag{2.2.4}$$

$$\sum_{j=0}^{2^\ell-1} \lambda_{\ell,j}\delta_{B_{\ell,j}} \in \mathscr{L}(\mathbb{R}^{n\times n}) \tag{2.2.5}$$

and

$$\left(B_{\ell,j}\right)_{\alpha,\alpha} = \begin{cases} k-1 & \textit{if } \alpha = \ell+1,\dots,n-i, \\ 0 & \textit{if } \alpha = n-i+1,\dots,n. \end{cases} \tag{2.2.6}$$

Moreover, if $B_{\ell,j} \notin E$ *then*

$$\left(B_{\ell,j}\right)_{\alpha,\alpha} \in \{0,k\}, \qquad \alpha = 1,\dots,\ell, \tag{2.2.7}$$

$$\lambda_{\ell,j} = \frac{(k-1)^{\beta_{\ell,j}}}{k^\ell}, \tag{2.2.8}$$

where

$$\beta_{\ell,j} := \operatorname{Card}\{\alpha \in \{1,\dots,\ell\} : \left(B_{\ell,j}\right)_{\alpha,\alpha} = k\} = \operatorname{rank}(B_{\ell,j}) - n + i + \ell,$$

and for each $B_{\ell,j'} \notin E$ *with* $j' \neq j$, *we have* $B_{\ell,j'} \neq B_{\ell,j}$.

Proof. We show the result by finite induction on ℓ. We define $B_{0,0} = A$ and $\lambda_{0,0} = 1$.

For $0 \leqslant \ell \leqslant n-i-1$, $0 \leqslant j \leqslant 2^\ell - 1$, we assume that $\{B_{\ell,j}\}_{j=0}^{2^\ell-1}$ and $\{\lambda_{\ell,j}\}_{j=0}^{2^\ell-1}$ have been defined, $B_{\ell,j}$ are diagonal, $\lambda_{\ell,j} \geqslant 0$ and equations (2.2.4)–(2.2.6) hold. We also assume that if $B_{\ell,j} \notin E$ then (2.2.7)–(2.2.8) hold, and for each $B_{\ell,j'} \notin E$ with $j' \neq j$, we have $B_{\ell,j'} \neq B_{\ell,j}$.

With the above induction hypotheses, we construct $\{B_{\ell+1,j}\}_{j=0}^{2^{\ell+1}-1}$ and $\{\lambda_{\ell+1,j}\}_{j=0}^{2^{\ell+1}-1}$ as follows. For any $0 \leqslant j \leqslant 2^\ell - 1$, we define

$$B_{\ell+1,2j} = \begin{cases} B_{\ell,j} - \operatorname{diag}\left(\underbrace{0,\dots,0}_{\ell}, k-1, \underbrace{0,\dots,0}_{n-\ell-1}\right), & \text{if } B_{\ell,j} \notin E, \\ B_{\ell,j}, & \text{if } B_{\ell,j} \in E, \end{cases}$$

$$B_{\ell+1,2j+1} = \begin{cases} B_{\ell,j} + \operatorname{diag}\left(\underbrace{0,\dots,0}_{\ell}, 1, \underbrace{0,\dots,0}_{n-\ell-1}\right), & \text{if } B_{\ell,j} \notin E, \\ B_{\ell,j}, & \text{if } B_{\ell,j} \in E, \end{cases}$$

$$\lambda_{\ell+1,2j} = \lambda_{\ell,j}\frac{1}{k} \quad \text{and} \quad \lambda_{\ell+1,2j+1} = \lambda_{\ell,j}\frac{k-1}{k}.$$

Now we check the induction hypotheses.

We have $B_{\ell,j} = \frac{1}{k}B_{\ell+1,2j} + \frac{k-1}{k}B_{\ell+1,2j+1}$ and $\operatorname{rank}(B_{\ell+1,2j} - B_{\ell+1,2j+1}) \leqslant 1$, so property (2.2.5) holds for $\ell+1$. Analogously, equalities (2.2.4) are easily seen to hold for $\ell+1$ as well.

Now, let $0 \leqslant j \leqslant 2^{\ell+1}-1$. We have

$$\left(B_{\ell+1,j}\right)_{\alpha,\alpha} = \left(B_{\ell,\lfloor\frac{j}{2}\rfloor}\right)_{\alpha,\alpha}, \qquad \alpha \neq \ell+1$$

and

$$\left(B_{\ell+1,j}\right)_{\ell+1,\ell+1} = \begin{cases} 0, & \text{if } B_{\ell,\lfloor\frac{j}{2}\rfloor} \notin E,\ j \text{ even,} \\ k, & \text{if } B_{\ell,\lfloor\frac{j}{2}\rfloor} \notin E,\ j \text{ odd,} \\ \left(B_{\ell,\lfloor\frac{j}{2}\rfloor}\right)_{\ell+1,\ell+1}, & \text{if } B_{\ell,\lfloor\frac{j}{2}\rfloor} \in E. \end{cases}$$

Therefore, equality (2.2.6) holds for $\ell+1$.

Now fix ℓ, j such that $B_{\ell+1,j} \notin E$. Then $B_{\ell,\lfloor\frac{j}{2}\rfloor} \notin E$ and, hence, property (2.2.7) holds for $\ell+1$. We also have

$$\beta_{\ell+1,j} = \begin{cases} \beta_{\ell,\frac{j}{2}}, & j \text{ even,} \\ \beta_{\ell,\frac{j-1}{2}} + 1, & j \text{ odd,} \end{cases}$$

so (2.2.8) holds for $\ell+1$. Finally, let $j' \neq j$ be such that $B_{\ell+1,j'} \notin E$. If $\lfloor\frac{j}{2}\rfloor \neq \lfloor\frac{j'}{2}\rfloor$, then $B_{\ell,\lfloor\frac{j'}{2}\rfloor} \neq B_{\ell,\lfloor\frac{j}{2}\rfloor}$, and, hence, $B_{\ell+1,j'} \neq B_{\ell+1,j}$, whereas if $\lfloor\frac{j}{2}\rfloor = \lfloor\frac{j'}{2}\rfloor$, then $(B_{\ell+1,j'})_{\ell+1,\ell+1} \neq (B_{\ell+1,j})_{\ell+1,\ell+1}$, and, hence, $B_{\ell+1,j'} \neq B_{\ell+1,j}$. □

Thanks to (2.2.6) and (2.2.7) we have, for all $0 \leqslant j \leqslant 2^{n-i}-1$,

$$B_{n-i,j} \in \bigcup_{\ell=i}^{n-m} S_\ell^k \cup E \tag{2.2.9}$$

whereas (2.2.8) yields

$$\lambda_{n-i,j} = \frac{(k-1)^{\operatorname{rank}(B_{n-i,j})}}{k^{n-i}}. \tag{2.2.10}$$

We define

$$\nu_A = \sum_{j=0}^{2^{n-i}-1} \lambda_{n-i,j}\delta_{B_{n-i,j}}, \tag{2.2.11}$$

which is a laminate due to (2.2.5).

From (2.2.9) we get

$$\nu_A\left(\bigcup_{\ell=0}^{i-1} S_\ell^k\right) = 0, \tag{2.2.12}$$

whereas for $\ell \in \{i,\dots,n-m\}$, we have, due to (2.2.6) and (2.2.10)

$$\nu_A\left(S^k_\ell\right) = \sum_{j:B_{n-i,j}\in S^k_\ell} \lambda_{n-i,j} = \sum_{j:B_{n-i,j}\in S^k_\ell} \frac{(k-1)^{n-\ell}}{k^{n-i}} \leqslant \binom{n-i}{n-\ell}\frac{(k-1)^{n-\ell}}{k^{n-i}}, \tag{2.2.13}$$

since the $B_{n-i,j}$ $(0 \leqslant j \leqslant 2^{n-i}-1)$ not in E are all different. Thus, for each $j \in \{1,\dots,N\}$, given A_j appearing in (2.2.2), we have constructed ν_{A_j} as in (2.2.11) if $A_j \notin E$ and $\nu_{A_j} = \delta_{A_j}$ if $A_j \in E$. So

$$\nu_{A_j}\left(S^k_\ell\right) = 0 \quad \text{if } A_j \in E, \qquad \forall \ell \in \{0,\dots,n-m\} \tag{2.2.14}$$

and we define the probability measure $\nu_k := \sum_{j=1}^N \lambda_j \nu_{A_j}$, which is supported in $\bigcup_{i=0}^{n-m} S^k_i \cup E$ by (2.2.9). In fact, ν_k is a laminate (by Corollary 1.0.3), but this is not important in the proof.

Claim 2. *Let $k \in \mathbb{N}$. Then*

$$|A| \leqslant k \textit{ for all } A \in \operatorname{supp}(\nu_k)\,. \tag{2.2.15}$$

Moreover, letting

$$C_k = \prod_{j=1}^{k-1}\left(1+2^n j^{-2}\right)$$

we have

$$\nu_k\left(S^k_i\right) \leqslant C_k k^{i-n}, \qquad i = 0,\dots,n-m \tag{2.2.16}$$

and

$$\nu_k\left(\bigcup_{i=0}^{n-m} S^k_i\right) \leqslant Ck^{-m} \tag{2.2.17}$$

for some $C > 0$ depending only on n.

Proof. Inequality (2.2.15) follows directly from the construction of ν_k.

In order to prove (2.2.16), we proceed by induction.

The inequality for $k = 1$ is immediate since $\nu_1 = \delta_I$. Let $k \geqslant 2$ and suppose that for $i = 0,\dots,n-m$, inequality (2.2.16) holds for $k-1$. Since the A_j of (2.2.2) are different, we have that $\nu_{k-1}(A_j) = \lambda_j$. Now, for all $\ell \in \{0,\dots,n-m\}$, we use (2.2.12) and (2.2.14) to get

$$\begin{aligned}\nu_k\left(S^k_\ell\right) &= \sum_{j=1}^N \nu_{k-1}\left(A_j\right)\nu_{A_j}\left(S^k_\ell\right) = \left[\sum_{j:A_j\in E} + \sum_{i=0}^{n-m}\sum_{j:A_j\in S^{k-1}_i}\right]\nu_{k-1}\left(A_j\right)\nu_{A_j}\left(S^k_\ell\right)\\ &= \sum_{i=0}^{\ell}\sum_{j:A_j\in S^{k-1}_i}\nu_{k-1}\left(A_j\right)\nu_{A_j}\left(S^k_\ell\right).\end{aligned}$$

We use that (2.2.16) is valid for $k-1$ to get that for all $i \in \{0,\dots,n-m\}$,

$$\frac{(k-1)^{n-\ell}}{k^{n-i}}\sum_{j:A_j\in S^{k-1}_i}\nu_{k-1}\left(A_j\right) = \frac{(k-1)^{n-\ell}}{k^{n-i}}\nu_{k-1}\left(S^{k-1}_i\right) \leqslant C_{k-1}\frac{(k-1)^{i-\ell}}{k^{n-i}}.$$

In addition,

$$\sum_{i=0}^{\ell}\binom{n-i}{n-\ell}((k-1)k)^{i-\ell}\leqslant 1+\sum_{i=0}^{\ell-1}\binom{n-i}{n-\ell}(k-1)^{2(i-\ell)}\leqslant 1+2^n(k-1)^{-2},$$

where we have used the crude inequality $\sum_{i=0}^{\ell-1}\binom{n-i}{n-\ell}\leqslant 2^n$. We combine the last three inequalities and (2.2.13) to get

$$\begin{aligned}\nu_k\left(S_\ell^k\right)&=\sum_{i=0}^{\ell}\sum_{j:A_j\in S_i^{k-1}}\nu_{k-1}\left(A_j\right)\nu_{A_j}\left(S_\ell^k\right)\leqslant\sum_{i=0}^{\ell}\sum_{j:A_j\in S_i^{k-1}}\nu_{k-1}\left(A_j\right)\binom{n-i}{n-\ell}\frac{(k-1)^{n-\ell}}{k^{n-i}}\\&\leqslant\sum_{i=0}^{\ell}\binom{n-i}{n-\ell}C_{k-1}\frac{(k-1)^{i-\ell}}{k^{n-i}}\leqslant k^{\ell-n}C_{k-1}\left(1+2^n(k-1)^{-2}\right)=C_kk^{\ell-n},\end{aligned}$$

which proves (2.2.16).

Finally, we observe that $\lim_{k\to\infty}C_k<\infty$. Consequently,

$$\nu_k\left(\bigcup_{i=0}^{n-m}S_i^k\right)\leqslant\sum_{i=0}^{n-m}C_kk^{i-n}\leqslant Ck^{-m}$$

for some $C>0$ depending only on n. □

Claim 3. *For all $k\in\mathbb{N}$,*

$$\nu_k\left(\{A\in\mathbb{R}^{n\times n}:|A|>t\}\right)\leqslant 2^mCt^{-m},\qquad t>0. \tag{2.2.18}$$

Proof. For simplicity of notation, the set $\{A\in\mathbb{R}^{n\times n}:|A|>t\}$ will be abbreviated as $\{|A|>t\}$.

We will prove the claim by induction on k. When $k=1$, we have $\nu_1=\delta_I$, hence, (2.2.18) is obvious. Now, we divide the inductive step in three cases, according to the values of t.

If $t\geqslant k+1$, we use (2.2.15) to obtain that $\nu_{k+1}(\{|A|>t\})=0$.

In the case $t<k$, we first show that if $|A|>t$, $A_j\in\operatorname{supp}(\nu_k)$ and $\nu_{A_j}(A)>0$, then $|A_j|>t$. Indeed, if $A_j\in E$, then $\nu_{A_j}=\delta_{A_j}$, so, as $\nu_{A_j}(A)>0$, we have $A=A_j$, and therefore $|A_j|>t$; whereas if $A_j\in\bigcup_{i=0}^{n-m}S_i^k$, then $|A_j|=k>t$. Therefore, if $|A_j|\leqslant t$, then $\nu_{A_j}(\{|A|>t\})=0$, and, hence,

$$\begin{aligned}\nu_{k+1}(\{|A|>t\})&=\sum_{\substack{j:A_j\in\operatorname{supp}(\nu_k)\\|A_j|>t}}\nu_k(A_j)\nu_{A_j}(\{|A|>t\})\\&\leqslant\nu_k\left(\{A_j\in\operatorname{supp}(\nu_k):|A_j|>t\}\right)\leqslant\nu_k(\{|A|>t\})\leqslant 2^mCt^{-m}.\end{aligned}$$

Finally, in the case $k\leqslant t<k+1$, we use that if $A\in\operatorname{supp}(\nu_{k+1})$ and $|A|>t$, then, by (2.2.15), we have that for all $A_j\in\operatorname{supp}(\nu_k)\cap E$, equalities $\nu_{A_j}(A)=\delta_{A_j}(A)=0$ are satisfied. Hence, thanks to (2.2.17),

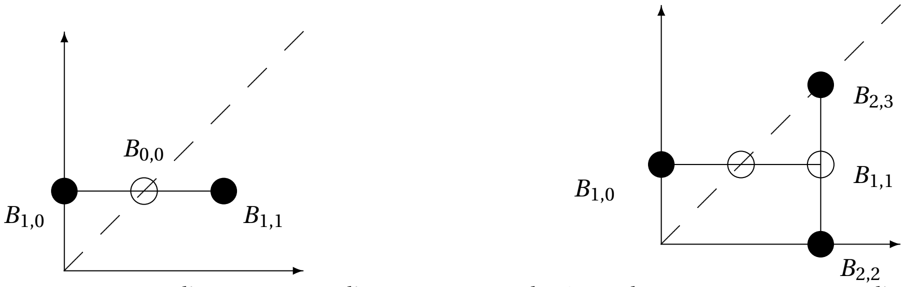
$$\nu_{k+1}(\{|A|>t\})=\sum_{\substack{j:A_j\in\operatorname{supp}(\nu_k)\\A_j\notin E}}\nu_k(A_j)\nu_{A_j}(\{|A|>t\})\leqslant\nu_k\left(\bigcup_{i=0}^{n-m}S_i^k\right)\leqslant Ck^{-m}\leqslant 2^mCt^{-m}.$$

Hence, estimate (2.2.18) is valid for $k+1$. This finishes the proof. □

Let ν be the weak* limit of ν_k as $k \to \infty$. Thanks to (2.2.17), ν is supported in E and, by (2.2.18), inequality (2.2.1) holds.

We will try to illustrate this construction with some pictures.

In the simplest case $n = m = 2$, E is the set of matrices with zero determinant, $\nu_1 = \delta_I$ and the first steps of the construction are depicted in Figure 2.1. In the first step (Figure 2.1(a)) we split $B_{0,0} = I$ into $B_{1,0} = \operatorname{diag}(0,1)$ and $B_{1,1} = \operatorname{diag}(2,1)$. As $B_{1,0} \in E$, the second step (Figure 2.1(b)) consists in splitting $B_{1,1}$ into $B_{2,2} = \operatorname{diag}(2,0)$ and $B_{2,3} = \operatorname{diag}(2,2)$.



(a) First: $B_{0,0} = I$, $B_{1,0} = \operatorname{diag}(0,1)$, $B_{1,1} = \operatorname{diag}(2,1)$.

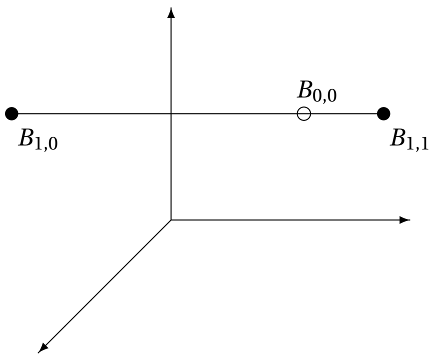
(b) Second: $B_{2,0} = B_{2,1} = B_{1,0} = \operatorname{diag}(0,1)$, $B_{2,2} = \operatorname{diag}(2,0)$, $B_{2,3} = \operatorname{diag}(2,2)$.

Figure 2.1: First and second splits for $n = m = 2$.

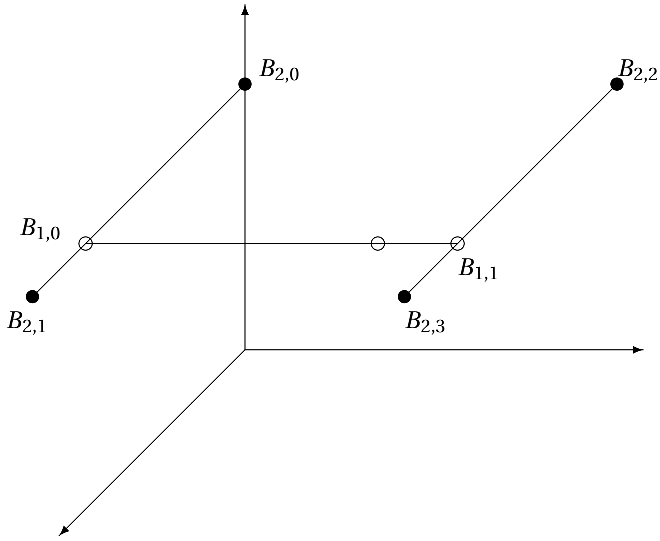
After the second split, we obtain

$$\nu_2 = \nu_I = \sum_{j=0}^{3} \lambda_{2,j} \delta_{B_{2,j}} = \frac{1}{2}\delta_{\operatorname{diag}(0,1)} + \frac{1}{4}\delta_{\operatorname{diag}(2,0)} + \frac{1}{4}\delta_{\operatorname{diag}(2,2)}.$$

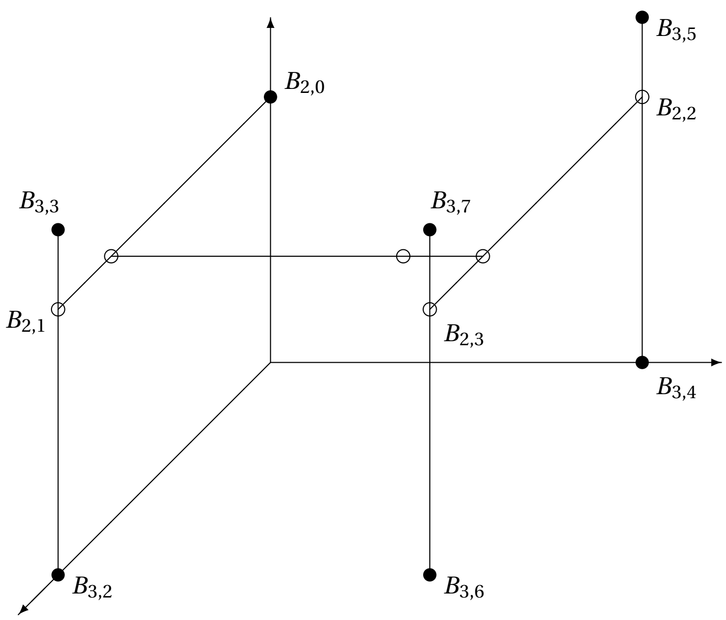
In the case $n = 3, m = 2$, E is the set of matrices of rank less than 2. In order to exemplify the passage from step $k-1$ to step k, if we start with a matrix in S_1^{k-1}, the construction is the same as in the case $n = m = 2$, whereas if we start with $A \in S_0^{k-1}$, we have $A = (k-1)I$ and Figure 2.2 shows the construction of ν_A.



(a) First: $B_{0,0} = (k-1)I$, $B_{1,0} = \operatorname{diag}(k-1,0,k-1)$, $B_{1,1} = \operatorname{diag}(k-1,k,k-1)$.



(b) Second: $B_{2,0} = \operatorname{diag}(0,0,k-1)$, $B_{2,1} = \operatorname{diag}(k,0,k-1)$, $B_{2,2} = \operatorname{diag}(0,k,k-1)$, $B_{2,3} = \operatorname{diag}(k,k,k-1)$.



(c) Third: $B_{3,0} = B_{3,1} = B_{2,0} = \operatorname{diag}(0,0,k-1)$, $B_{3,2} = \operatorname{diag}(k,0,0)$, $B_{3,3} = \operatorname{diag}(k,0,k)$, $B_{3,4} = \operatorname{diag}(0,k,0)$, $B_{3,5} = \operatorname{diag}(0,k,k)$, $B_{3,6} = \operatorname{diag}(k,k,0)$, $B_{3,7} = kI$.

Figure 2.2: First, second and thrid splits for $n = 3$, $m = 2$.

We get at the end

$$\begin{aligned}\nu_A = \sum_{j=0}^{7} \lambda_{3,j}\delta_{B_{3,j}} &= \frac{1}{k^2}\delta_{\mathrm{diag}(0,0,k-1)} + \frac{(k-1)^2}{k^3}\delta_{\mathrm{diag}(k,0,k)} + \frac{k-1}{k^3}\delta_{\mathrm{diag}(k,0,0)} \\ &+ \frac{(k-1)^2}{k^3}\delta_{\mathrm{diag}(0,k,k)} + \frac{k-1}{k^3}\delta_{\mathrm{diag}(0,k,0)} + \frac{(k-1)^2}{k^3}\delta_{\mathrm{diag}(k,k,0)} + \frac{(k-1)^3}{k^3}\delta_{kI}.\end{aligned}$$

The construction of ν_k would entail the analogous construction for each $A \in S_0^{k-1} \cup S_1^{k-1}$.

2.2.2 Construction of the laminate and its approximation

This subsection constructs the sequence of laminates together with their approximations by functions. We will continuously use the following sets and constants.

For $j \in \mathbb{N}$, we define the sequence of open sets E_j by

$$E_j = \{A \in \Gamma_+ : |A| > \frac{1}{2} + 2^{-j},\ 2^{-j-m} < \sigma_i(A)\max\{|A|^{m-1}, 1\} < 2^{-j} \ \text{ for } 1 \leqslant i \leqslant n-m+1\}.$$

For $a \in \mathbb{N}$, $0 \leqslant a \leqslant n-m$, $j \in \mathbb{N}$ and $\mathscr{R} > \frac{1}{2} + 2^{-j}$ such that

$$\rho_{j,\mathscr{R}} := \frac{3 \cdot 2^{-j-2}}{\max\{\mathscr{R}^{m-1}, 1\}} < \mathscr{R},$$

we define the closed sets

$$E^a_{j,\mathscr{R}} = \{A \in \Gamma_+ : \sigma_i(A) = \rho_{j,\mathscr{R}} \text{ for } 1 \leqslant i \leqslant a,\ \ \sigma_i(A) = \mathscr{R} \text{ for } a+1 \leqslant i \leqslant n\}.$$

We also denote

$$E^a_j = \bigcup_{\mathscr{R} \in (\frac{1}{2}+2^{-j}, \infty)} E^a_{j,\mathscr{R}}.$$

The sets E_j approximate the set of positive semidefinite matrices with rank less than m, and the sets $E^a_{j,\mathscr{R}}$ approximate the set

$$\{\mathscr{R}\, Q\, I_a\, Q^T : Q \in SO(n)\}, \qquad I_a = \mathrm{diag}(\underbrace{0,\ldots,0}_{a}, \underbrace{1\ldots,1}_{n-a}).$$

The number $\mathscr{R}$ plays the role of k in Subsection 2.2.1 and eventually will tend to infinity. The number $\rho_{j,\mathscr{R}}$ will tend to zero: the reason why it appears in the definition of $E^a_{j,\mathscr{R}}$ is that, even though $E^a_{j,\mathscr{R}}$ approximates a subset of matrices of rank $n-a$, we need them to be invertible.

Given $j \in \mathbb{N}$ and $\mathscr{R} > \frac{1}{2} + 2^{-j}$ we define

$$\begin{aligned}r_{j,\mathscr{R}} := \frac{1}{2}\min\Big\{&1 - \left(\frac{2}{3}\right)^{\frac{1}{m-1}}, \rho_{j,\mathscr{R}}\left(1 - \frac{\max\{1, \mathscr{R}^{m-1}\}}{(\mathscr{R}+1)^{m-1}}\right), \frac{\rho_{j,\mathscr{R}}}{3}, \mathscr{R} - \frac{1}{2} - 2^{-j}, \\ &\frac{1}{2^n}(\mathscr{R}+1)^{-m-1}, \frac{1-\rho_{j,\mathscr{R}+1}}{\max\{1,\mathscr{R}\}}\Big\}.\end{aligned} \tag{2.2.19}$$

Note that $r_{j,\mathscr{R}}$ is just a sufficiently small positive constant depending on m, j, $\mathscr{R}$ (and, hence, on $\rho_{j,\mathscr{R}}$): it will play the role of the δ of Proposition 1.3.5. We prefer to write its exact expression to make it easier to follow a series of inequalities involving it.

For $j \in \mathbb{N}$, $a_0, a \in \{0,\dots,n-m\}$, $\mathscr{R} > \rho_{j,\mathscr{R}}$ we denote

$$\begin{aligned} C(j,\mathscr{R},a_0,a) = &\sum_{b=\max\{0,a_0+a-n\}}^{\min\{a_0,a\}} \binom{a_0}{b}\binom{n-a_0}{a-b}\left(\frac{\mathscr{R}+r_{j,\mathscr{R}}}{\mathscr{R}+1}\right)^{n-a_0-a+b} \\ &\times \left(\frac{1}{\max\{1,\mathscr{R}\}}\right)^{a-b}\left(\frac{2^{-j}}{(\mathscr{R}+1)\max\{1,\mathscr{R}^{m-1}\}}\right)^{a_0-b}. \end{aligned} \tag{2.2.20}$$

The next lemma constructs a laminate with the required integrability. The second part of its proof follows that of Subsection 2.2.1.

Lemma 2.2.2. *Let $j \in \mathbb{N}$, $a_0 \in \{0,\dots,n-m\}$, $\mathscr{R} > 0$ with $\rho_{j,\mathscr{R}} < \mathscr{R}$ and $A \in \Gamma_+$ be such that $\operatorname{dist}(A, E^{a_0}_{j,\mathscr{R}}) < r_{j,\mathscr{R}}$. Then there exists $\nu \in \mathscr{L}(\mathbb{R}^{n\times n})$ such that $\overline{\nu} = A$,*

$$\operatorname{supp}\nu \subset \left(\bigcup_{a=0}^{n-m} E^a_{j,\mathscr{R}+1} \cup E_j\right) \cap \{\xi \in \mathbb{R}^{n\times n} : |\xi| \leqslant \mathscr{R}+1\}$$

and for $0 \leqslant a \leqslant n-m$,

$$\nu\left(E^a_{j,\mathscr{R}+1}\right) \leqslant C(j,\mathscr{R},a_0,a).$$

Proof. There exist $Q \in SO(n)$ and $B \in E^{a_0}_{j,\mathscr{R}}$ such that $A = Q \operatorname{diag}(\sigma_1,\dots,\sigma_n)\, Q^T$, with $0 < \sigma_1 \leqslant \dots \leqslant \sigma_n$ and $|A-B| < r_{j,\mathscr{R}}$. Using the inequality

$$|\sigma_i(A) - \sigma_i(B)| \leqslant |A-B|, \qquad i = 1,\dots,n, \tag{2.2.21}$$

(see, e.g., [66, Corollary 4.5]), we find that

$$|\sigma_i - \rho_{j,\mathscr{R}}| < r_{j,\mathscr{R}} \quad \text{for } 1 \leqslant i \leqslant a_0, \quad \text{and} \quad |\sigma_i - \mathscr{R}| < r_{j,\mathscr{R}} \quad \text{for } a_0+1 \leqslant i \leqslant n. \tag{2.2.22}$$

In order to construct the desired laminate we prove that:

1) $\sigma_n < \mathscr{R}+1$.

2) $\rho_{j,\mathscr{R}+1} < \sigma_1$.

3) $2^{-j-m} < \rho_{j,\mathscr{R}+1} \max\left\{1, \sigma_n^{m-1}\right\} < 2^{-j}$.

Inequality 1) is obvious thanks to (2.2.22) since $r_{j,\mathscr{R}} < 1$. By (2.2.22), the definition of $\rho_{j,\mathscr{R}+1}$ and (2.2.19) we obtain

$$\rho_{j,\mathscr{R}+1} = \frac{3\cdot 2^{-j-2}}{(\mathscr{R}+1)^{m-1}} = \frac{\rho_{j,\mathscr{R}} \max\{1,\mathscr{R}^{m-1}\}}{(\mathscr{R}+1)^{m-1}} < \rho_{j,\mathscr{R}} - r_{j,\mathscr{R}} < \sigma_1,$$

where in the last inequality we have differentiated the cases $a_0 = 0$ and $a_0 > 0$. So we have 2). Lastly, we prove 3). On the one hand,

$$\rho_{j,\mathscr{R}+1} \max\left\{1, \sigma_n^{m-1}\right\} \leqslant \rho_{j,\mathscr{R}+1} \max\left\{1, (\mathscr{R}+r_{j,\mathscr{R}})^{m-1}\right\} \leqslant \rho_{j,\mathscr{R}+1}(\mathscr{R}+1)^{m-1} = 3\cdot 2^{-j-2} < 2^{-j}$$

and, on the other hand,

$$\rho_{j,\mathscr{R}+1}\max\left\{1,\sigma_n^{m-1}\right\} \geqslant \rho_{j,\mathscr{R}+1}\max\left\{1,(\mathscr{R}-r_{j,\mathscr{R}})^{m-1}\right\} = \frac{3\cdot 2^{-j-2}\max\left\{1,(\mathscr{R}-r_{j,\mathscr{R}})^{m-1}\right\}}{(\mathscr{R}+1)^{m-1}}.$$

Therefore, if $\mathscr{R}\leqslant 1$ we have

$$\frac{3\cdot 2^{-j-2}\max\left\{1,(\mathscr{R}-r_{j,\mathscr{R}})^{m-1}\right\}}{(\mathscr{R}+1)^{m-1}} \geqslant 3\cdot 2^{-j-m-1} > 2^{-j-m},$$

whereas if $\mathscr{R}>1$ we use (2.2.19) to obtain

$$3\cdot 2^{-j-2}\frac{\max\left\{1,(\mathscr{R}-r_{j,\mathscr{R}})^{m-1}\right\}}{(\mathscr{R}+1)^{m-1}} \geqslant 3\cdot 2^{-j-2}\frac{(\mathscr{R}-r_{j,\mathscr{R}})^{m-1}}{(\mathscr{R}+1)^{m-1}} \geqslant 3\cdot 2^{-j-m-1}(1-r_{j,\mathscr{R}})^{m-1} > 2^{-j-m}.$$

Thus, 3) is proved.

Now we build the laminate, following the lines of Subsection 2.2.1. We shall construct families

$$\{B_{\ell,i}\}_{\substack{\ell=0,\dots,n\\ i=0,\dots 2^\ell-1}} \subset \Gamma_+ \quad \text{and} \quad \{\lambda_{\ell,i}\}_{\substack{\ell=0,\dots,n\\ i=0,\dots 2^\ell-1}} \subset [0,1] \tag{2.2.23}$$

by finite induction on ℓ. Let $B_{0,0}=A$, $\lambda_{0,0}=1$ and for $0\leqslant\ell\leqslant n-1$, $0\leqslant i\leqslant 2^\ell-1$, we assume $\{B_{\ell,i}\}_{i=0}^{2^\ell-1}$ and $\{\lambda_{\ell,i}\}_{i=0}^{2^\ell-1}$ have been defined, $Q^TB_{\ell,j}Q$ is diagonal, $\lambda_{\ell,i}\geqslant 0$,

$$\sum_{i=0}^{2^\ell-1}\lambda_{\ell,i}=1, \qquad B_{0,0}=\sum_{i=0}^{2^\ell-1}\lambda_{\ell,i}B_{\ell,i}, \qquad \sum_{i=0}^{2^\ell-1}\lambda_{\ell,i}\delta_{B_{\ell,i}}\in\mathscr{L}(\mathbb{R}^{n\times n}) \tag{2.2.24}$$

and

$$\left(Q^TB_{\ell,i}Q\right)_{\alpha,\alpha}=\sigma_\alpha \quad \text{if } \alpha=\ell+1,\dots,n. \tag{2.2.25}$$

We also assume that if $B_{\ell,i}\notin E_j$ then

$$\left(Q^TB_{\ell,i}Q\right)_{\alpha,\alpha}\in\{\rho_{j,\mathscr{R}+1},\mathscr{R}+1\}, \qquad \alpha=1,\dots,\ell, \tag{2.2.26}$$

and when we let

$$\beta_{\ell,i} := \operatorname{Card}\{\alpha\in\{1,\dots,\min\{a_0,\ell\}\} : \left(Q^TB_{\ell,i}Q\right)_{\alpha,\alpha}=\rho_{j,\mathscr{R}+1}\},$$

$$\gamma_{\ell,i} := \operatorname{Card}\{\alpha\in\{a_0+1,\dots,\ell\} : \left(Q^TB_{\ell,i}Q\right)_{\alpha,\alpha}=\rho_{j,\mathscr{R}+1}\},$$

then

$$\beta_{\ell,i}+\gamma_{\ell,i}\leqslant n-m, \tag{2.2.27}$$

and, calling

$$U:=\frac{2^{-j}}{(\mathscr{R}+1)\max\{1,\mathscr{R}^{m-1}\}}, \qquad V:=\frac{1}{\max\{1,\mathscr{R}\}}, \qquad W:=\frac{\mathscr{R}+r_{j,\mathscr{R}}}{\mathscr{R}+1}, \tag{2.2.28}$$

we have

$$\lambda_{\ell,i} \leqslant U^{\min\{a_0,\ell\}-\beta_{\ell,i}}\, V^{\gamma_{\ell,i}}\, W^{\max\{0,\ell-a_0-\gamma_{\ell,i}\}}. \tag{2.2.29}$$

We assume additionally that for each $B_{\ell,i'} \notin E_j$ such that $i' \neq i$, we have $B_{\ell,i'} \neq B_{\ell,i}$.

With the above induction hypotheses, we construct $\{B_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ and $\{\lambda_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ as follows. For any $0 \leqslant i \leqslant 2^\ell - 1$, define

$$B_{\ell+1,2i} = \begin{cases} B_{\ell,i} - Q\operatorname{diag}\left(\underbrace{0,\dots,0}_{\ell}, \sigma_{\ell+1} - \rho_{j,\mathscr{R}+1}, \underbrace{0,\dots,0}_{n-\ell-1}\right)Q^T, & \text{if } B_{\ell,i} \notin E_j, \\ B_{\ell,i}, & \text{if } B_{\ell,i} \in E_j, \end{cases}$$

$$B_{\ell+1,2i+1} = \begin{cases} B_{\ell,i} + Q\operatorname{diag}\left(\underbrace{0,\dots,0}_{\ell}, \mathscr{R} + 1 - \sigma_{\ell+1}, \underbrace{0,\dots,0}_{n-\ell-1}\right)Q^T, & \text{if } B_{\ell,i} \notin E_j, \\ B_{\ell,i}, & \text{if } B_{\ell,i} \in E_j, \end{cases}$$

$$\lambda_{\ell+1,2i} = \lambda_{\ell,i}\frac{\mathscr{R}+1-\sigma_{\ell+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \quad \text{and} \quad \lambda_{\ell+1,2i+1} = \lambda_{\ell,i}\frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}}.$$

So $\operatorname{rank}\left(B_{\ell+1,2i} - B_{\ell+1,2i+1}\right) \leqslant 1$, $\lambda_{\ell+1,2i} \geqslant 0$ by 1), $\lambda_{\ell+1,2i+1} \geqslant 0$ by 2), and

$$B_{\ell,i} = \frac{\mathscr{R}+1-\sigma_{\ell+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}}B_{\ell+1,2i} + \frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}}B_{\ell+1,2i+1}.$$

With this, we can easily see that properties (2.2.24) hold for $\ell+1$. In what follows, $0 \leqslant i \leqslant 2^{\ell+1}-1$. We have

$$\left(Q^T B_{\ell+1,i} Q\right)_{\alpha,\alpha} = \left(Q^T B_{\ell,\lfloor \frac{i}{2}\rfloor} Q\right)_{\alpha,\alpha}, \qquad \alpha \neq \ell+1,$$

$$\left(Q^T B_{\ell+1,i} Q\right)_{\ell+1,\ell+1} = \begin{cases} \rho_{j,\mathscr{R}+1}, & \text{if } B_{\ell,\lfloor \frac{i}{2}\rfloor} \notin E_j, i \text{ even}, \\ \mathscr{R}+1, & \text{if } B_{\ell,\lfloor \frac{i}{2}\rfloor} \notin E_j, i \text{ odd}, \\ \left(Q^T B_{\ell,\lfloor \frac{i}{2}\rfloor} Q\right)_{\ell+1,\ell+1}, & \text{if } B_{\ell,\lfloor \frac{i}{2}\rfloor} \in E_j. \end{cases}$$

Therefore, property (2.2.25) holds for $\ell+1$. Now fix ℓ, i such that $B_{\ell+1,i} \notin E_j$. Then $B_{\ell+1,\lfloor \frac{i}{2}\rfloor} \notin E_j$, property (2.2.26) holds for $\ell+1$, and

$$\beta_{\ell+1,i} = \begin{cases} \beta_{\ell,\frac{i}{2}} + 1 & \text{if } i \text{ is even}, \ell < a_0, \\ \beta_{\ell,\frac{i}{2}} & \text{if } i \text{ is even}, \ell \geqslant a_0, \\ \beta_{\ell,\frac{i-1}{2}} & \text{if } i \text{ is odd}, \end{cases} \qquad \gamma_{\ell+1,i} = \begin{cases} \gamma_{\ell,\frac{i}{2}} & \text{if } i \text{ is even}, \ell < a_0, \\ \gamma_{\ell,\frac{i}{2}} + 1 & \text{if } i \text{ is even}, \ell \geqslant a_0, \\ \gamma_{\ell,\frac{i-1}{2}} & \text{if } i \text{ is odd}. \end{cases}$$

Using (2.2.27) we find that $\beta_{\ell+1,i} + \gamma_{\ell+1,i} \leqslant n - m + 1$. On the other hand, we have shown that

$$\sigma_\alpha(B_{\ell+1,i}) \in \{\rho_{j,\mathscr{R}+1}, \mathscr{R}+1, \sigma_{\ell+2}, \dots, \sigma_n\}, \qquad \alpha = 1, \dots, n.$$

Thus, if we had $\beta_{\ell+1,i} + \gamma_{\ell+1,i} = n - m + 1$ then, by 1) and 2) we would get

$$\sigma_\alpha(B_{\ell+1,i}) = \rho_{j,\mathscr{R}+1}, \qquad \alpha = 1, \dots, n-m+1$$

and by 3), $B_{\ell+1,i} \in E_j$, which is a contradiction. Therefore, (2.2.27) holds for $\ell+1$.

Now let $i' \neq i$ be such that $B_{\ell+1,i'} \notin E_j$. If $\lfloor \frac{i}{2} \rfloor \neq \lfloor \frac{i'}{2} \rfloor$, then $B_{\ell,\lfloor \frac{i'}{2} \rfloor} \neq B_{\ell,\lfloor \frac{i}{2} \rfloor}$, and, hence, $B_{\ell+1,i'} \neq B_{\ell+1,i}$, whereas if $\lfloor \frac{i}{2} \rfloor = \lfloor \frac{i'}{2} \rfloor$, then $(B_{\ell+1,i'})_{\ell+1,\ell+1} \neq (B_{\ell+1,j})_{\ell+1,\ell+1}$, and, hence, $B_{\ell+1,i'} \neq B_{\ell+1,i}$.

Now we bound $\lambda_{\ell+1,i}$. Recall the notation (2.2.28) and the induction hypothesis (2.2.29). If i is even and $\ell < a_0$, we have $\max\{0, \ell+1-a_0-\gamma_{\ell,i}\} = 0$ and, therefore,

$$\begin{aligned}\lambda_{\ell+1,i} &= \lambda_{\ell,\frac{i}{2}} \frac{\mathscr{R}+1-\sigma_{\ell+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \lambda_{\ell,\frac{i}{2}} \leqslant U^{\min\{a_0,\ell\}-\beta_{\ell,\frac{i}{2}}}\, V^{\gamma_{\ell,\frac{i}{2}}}\, W^{\max\{0,\ell-a_0-\gamma_{\ell,\frac{i}{2}}\}} \\ &= U^{\min\{a_0,\ell+1\}-\beta_{\ell+1,i}}\, V^{\gamma_{\ell+1,i}}\, W^{\max\{0,\ell+1-a_0-\gamma_{\ell+1,i}\}}.\end{aligned}$$

If i is even and $\ell \geqslant a_0$, using (2.2.22) and (2.2.19), we have

$$\frac{\mathscr{R}+1-\sigma_{\ell+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{1+r_{j,\mathscr{R}}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{1}{\max\{1,\mathscr{R}\}},$$

therefore

$$\begin{aligned}\lambda_{\ell+1,i} &= \lambda_{\ell,\frac{i}{2}} \frac{\mathscr{R}+1-\sigma_{\ell+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \lambda_{\ell,\frac{i}{2}} \frac{1}{\max\{1,\mathscr{R}\}} \leqslant U^{\min\{a_0,\ell\}-\beta_{\ell,\frac{i}{2}}}\, V^{\gamma_{\ell,\frac{i}{2}}+1}\, W^{\max\{0,\ell-a_0-\gamma_{\ell,\frac{i}{2}}\}} \\ &= U^{\min\{a_0,\ell+1\}-\beta_{\ell+1,i}}\, V^{\gamma_{\ell+1,i}}\, W^{\max\{0,\ell+1-a_0-\gamma_{\ell+1,i}\}}.\end{aligned}$$

If i is odd and $\ell < a_0$, then, by (2.2.22) and the definition of $r_{j,\mathscr{R}}$ and $\rho_{j,\mathscr{R}}$, we have

$$\frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{\rho_{j,\mathscr{R}}+r_{j,\mathscr{R}}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{\rho_{j,\mathscr{R}}+r_{j,\mathscr{R}}}{\mathscr{R}+1} \leqslant \frac{4\rho_{j,\mathscr{R}}}{3(\mathscr{R}+1)} \leqslant \frac{2^{-j}}{(\mathscr{R}+1)\max\{1,\mathscr{R}^{m-1}\}}$$

and

$$\begin{aligned}\lambda_{\ell+1,i} &= \lambda_{\ell,\frac{i-1}{2}} \frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \lambda_{\ell,\frac{i-1}{2}} U \leqslant U^{\min\{a_0,\ell\}-\beta_{\ell,\frac{i-1}{2}}+1}\, V^{\gamma_{\ell,\frac{i-1}{2}}}\, W^{\max\{0,\ell-a_0-\gamma_{\ell,\frac{i-1}{2}}\}} \\ &= U^{\min\{a_0,\ell+1\}-\beta_{\ell+1,i}}\, V^{\gamma_{\ell+1,i}}\, W^{\max\{0,\ell+1-a_0-\gamma_{\ell+1,i}\}}.\end{aligned}$$

Finally, if i is odd and $\ell \geqslant a_0$ we have $\gamma_{\ell,i} \leqslant \ell - a_0$ for all $i = 0,\dots,2^\ell-1$, and

$$\frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{\mathscr{R}+r_{j,\mathscr{R}}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \frac{\mathscr{R}+r_{j,\mathscr{R}}}{\mathscr{R}+1},$$

so

$$\begin{aligned}\lambda_{\ell+1,i} &= \lambda_{\ell,\frac{i-1}{2}} \frac{\sigma_{\ell+1}-\rho_{j,\mathscr{R}+1}}{\mathscr{R}+1-\rho_{j,\mathscr{R}+1}} \leqslant \lambda_{\ell,\frac{i-1}{2}} W \leqslant U^{\min\{a_0,\ell\}-\beta_{\ell,\frac{i-1}{2}}}\, V^{\gamma_{\ell,\frac{i-1}{2}}}\, W^{\max\{0,\ell-a_0-\gamma_{\ell,\frac{i-1}{2}}\}+1} \\ &= U^{\min\{a_0,\ell+1\}-\beta_{\ell+1,i}}\, V^{\gamma_{\ell+1,i}}\, W^{\max\{0,\ell+1-a_0-\gamma_{\ell+1,i}\}}.\end{aligned}$$

With this, we finish the inductive construction of the families (2.2.23). In particular, for all $0 \leqslant i \leqslant 2^n-1$, if $B_{n,i} \notin E_j$ we have

$$\lambda_{n,i} \leqslant U^{a_0-\beta_{n,i}}\, V^{\gamma_{n,i}}\, W^{n-a_0-\gamma_{n,i}}, \tag{2.2.30}$$

$$\left(Q^T B_{n,i} Q\right)_{\alpha,\alpha} \in \{\rho_{j,\mathscr{R}+1}, \mathscr{R}+1\}, \qquad \alpha = 1,\dots,n,$$

and

$$a := \operatorname{Card}\{\alpha : (Q^T B_{n,i} Q)_{\alpha,\alpha} = \rho_{j,\mathscr{R}+1}\} = \beta_{n,i} + \gamma_{n,i} \leqslant n-m.$$

Therefore, by definition of $E^a_{j,\mathscr{R}+1}$, we get $B_{n,i} \in E^a_{j,\mathscr{R}+1}$. Hence, for all $0 \leqslant i \leqslant 2^n - 1$, we have proved that

$$B_{n,i} \in \bigcup_{a=0}^{n-m} E^a_{j,\mathscr{R}+1} \cup E_j.$$

We define

$$\nu = \sum_{i=0}^{2^n-1} \lambda_{n,i} \delta_{B_{n,i}},$$

which is a laminate by (2.2.24).

In order to estimate $\nu(E^a_{j,\mathscr{R}+1})$, we observe that for $B_{n,i} \in E^a_{j,\mathscr{R}+1}$ we have $\max\{0, a_0 + a - n\} \leqslant \beta_{n,i} \leqslant \min\{a_0, a\}$. Therefore

$$\begin{aligned}
\nu\left(E^a_{j,\mathscr{R}+1}\right) &= \sum_{\substack{i:\beta_{n,i}+\gamma_{n,i}=a \\ B_{n,i}\in E^a_{j,\mathscr{R}+1}}} \lambda_{n,i} = \sum_{b=\max\{0,a_0+a-n\}}^{\min\{a_0,a\}} \sum_{\substack{i:\beta_{n,i}=b,\gamma_{n,i}=a-b \\ B_{n,i}\in E^a_{j,\mathscr{R}+1}}} \lambda_{n,i} \\
&\leqslant \sum_{b=\max\{0,a_0+a-n\}}^{\min\{a_0,a\}} \binom{a_0}{b}\binom{n-a_0}{a-b} U^{a_0-b} V^{a-b} W^{n-a_0-a+b},
\end{aligned}$$

where we have used that the $B_{n,i}$ $(0 \leqslant i \leqslant 2^n - 1)$ in $E^a_{j,\mathscr{R}+1}$ are all different, as well as estimate (2.2.30). This concludes the proof. □

The following result constructs a function whose gradient approximates the laminate of the previous lemma and have the desired integrability.

Lemma 2.2.3. *Let $\alpha \in (0,1)$ and $\delta > 0$. Then there is a $j_1 \in \mathbb{N}$ such that for any $j \geqslant j_1$, any bounded open set $\omega \subset \mathbb{R}^n$ and any $F \in \Gamma_+$ such that $\operatorname{dist}(F, \bigcup_{a=0}^{n-m} E^a_{j,|F|}) < r_{j,|F|}$, there exists a piecewise affine homeomorphism $f \in W^{1,1}(\omega, F\omega) \cap C^\alpha(\overline{\omega}, \overline{F\omega})$ such that*

i) $f(x) = Fx$ *for all* $x \in \partial\omega$,

ii) $\|f - F\|_{C^\alpha(\overline{\omega})} < \delta$ *and* $\|f^{-1} - F^{-1}\|_{C^\alpha(\overline{F\omega})} < \delta$,

iii) $Df(x) \in E_j$ *a.e.* $x \in \omega$,

iv) for all $t > 0$,

$$\frac{|\{x \in \omega : |Df(x)| > t\}|}{|\omega|} \lesssim |F|^m t^{-m}.$$

Proof. Let $\mathscr{R} = |F|$, $a_0 \in \{0,\dots,n-m\}$, $Q \in SO(n)$ and

$$A = Q \operatorname{diag}\Big(\underbrace{\rho_{j,\mathscr{R}},\dots,\rho_{j,\mathscr{R}}}_{a_0}, \underbrace{\mathscr{R},\dots,\mathscr{R}}_{n-a_0}\Big) Q^T \in E^{a_0}_{j,\mathscr{R}}$$

be such that $|F-A|<r_{j,\mathscr{R}}$, and for $a=0,\dots,n-m$, define the sets

$$S^a_{j,\mathscr{R}} := \left\{M\in\Gamma_+ : \operatorname{dist}\left(M,E^a_{j,\mathscr{R}}\right) < r_{j,\mathscr{R}}\right\}.$$

Note that the sets $S^0_{j,\mathscr{R}},\dots,S^{n-m}_{j,\mathscr{R}}$ are pairwise disjoint. Indeed, if $S^{a_1}_{j,\mathscr{R}}\cap S^{a_2}_{j,\mathscr{R}}\neq\emptyset$ for some $a_1\neq a_2$ we would obtain, thanks to inequality (2.2.21), $|\mathscr{R}-\rho_{j,\mathscr{R}}|<2r_{j,\mathscr{R}}$, which contradicts the definition of $r_{j,\mathscr{R}}$.

Given $k\in\mathbb{N}$, we define $\tilde{k}=k+\mathscr{R}-1$. We will construct by induction a sequence $\{f_k\}_{k\in\mathbb{N}}$ of piecewise affine homeomorphisms such that

(a) $f_k(x)=Fx$ for all $x\in\partial\omega$.

(b) $\|f_k-f_{k-1}\|_{C^\alpha(\overline{\omega})}<2^{-k}\delta$ and $\|f_k^{-1}-f_{k-1}^{-1}\|_{C^\alpha(\overline{F\omega})}<2^{-k}\delta$.

(c) $Df_k(x)\in E_j\cup\bigcup_{a=0}^{n-m}S^a_{j,\tilde{k}}$ for a.e. $x\in\omega$.

(d) $\left|Df_k\right|<\tilde{k}+1$ in $\omega\setminus\omega_k$, with $\omega_k:=\bigcup_{a=0}^{n-m}\omega^a_k$ and

$$\omega^a_k:=\{x\in\omega : f_k \text{ is affine in a neighbourhood of } x \text{ and } Df_k(x)\in S^a_{j,\tilde{k}}\},\quad 0\leqslant a\leqslant n-m.$$

(e) There exists $j_1\in\mathbb{N}$ such that for any $j\geqslant j_1$ we have

$$\frac{|\omega^a_k|}{|\omega|}\lesssim \tilde{k}^{\frac{1}{2}+a-n}\quad\text{for } 0\leqslant a\leqslant n-m-1,\quad\text{and}\quad \frac{|\omega^{n-m}_k|}{|\omega|}\lesssim\tilde{k}^{-m}\sum_{d=1}^{k}\tilde{d}^{-\frac{4}{3}}.$$

(f) $\omega_k\supset\omega_{k+1}$ and $f_{k+1}|_{\omega\setminus\omega_k}=f_k|_{\omega\setminus\omega_k}$.

Note that the sets ω^a_k defined in *(d)* are open and pairwise disjoint, since so are $S^a_{j,\mathscr{R}}$. Recall also that $\tilde{d}$ stands for $d+\mathscr{R}-1$.

For $k=0,1$ we see that the choices $f_0(x)=f_1(x)=Fx$, $\omega^{a_0}_0=\omega^{a_0}_1=\omega$ and $\omega^a_0=\omega^a_1=\emptyset$ for $a\neq a_0$ satisfy all the assumptions.

Fix $k\in\mathbb{N}$ and assume f_k has been constructed. We obtain f_{k+1} by modifying f_k on the sets ω^a_k. Since f_k is piecewise affine, there exists a family $\{\omega_i\}_{i\in\mathbb{N}}\subset\omega$ of pairwise disjoint open sets such that $|\omega\setminus\bigcup_{i\in\mathbb{N}}\omega_i|=0$ and $f|_{\omega_i}$ is affine for each $i\in\mathbb{N}$. More precisely, fix $a\in\{0,\dots,n-m\}$ and define $\omega^a_{k,i}:=\omega_i\cap\omega^a_k$ for each $i\in\mathbb{N}$, which is an open set. From now on, we only deal with those $\omega^a_{k,i}$ that are non-empty. Then there exist families $\{A^a_{k,i}\}_{i\in\mathbb{N}}\subset S^a_{j,\tilde{k}}$ and $\{b^a_{k,i}\}_{i\in\mathbb{N}}\subset\mathbb{R}^n$ such that $f_k(x)=A^a_{k,i}x+b^a_{k,i}$ for $x\in\omega^a_{k,i}$.

Let $\nu_{A^a_{k,i}}$ be the laminate of Lemma 2.2.2 that satisfies $\overline{\nu_{A^a_{k,i}}}=A^a_{k,i}$,

$$\operatorname{supp}\nu_{A^a_{k,i}}\subset\left(\bigcup_{b=0}^{n-m}E^b_{j,\tilde{k}+1}\cup E_j\right)\cap\{\xi\in\mathbb{R}^{n\times n}:|\xi|\leqslant\tilde{k}+1\}$$

and for $0\leqslant b\leqslant n-m$,

$$\nu_{A^a_{k,i}}\left(E^b_{j,\tilde{k}+1}\right)\leqslant C(j,\tilde{k},a,b).$$

We apply Proposition 1.3.5 to that laminate and obtain a piecewise affine homeomorphism $g^a_{k,i}:\omega^a_{k,i}\to A^a_{k,i}\omega^a_{k,i}+b^a_{k,i}$ with *(g)* $g^a_{k,i}(x) = A^a_{k,i}x + b^a_{k,i}$ on $\partial\omega^a_{k,i}$.

(h) $|Dg^a_{k,i}(x)| < \tilde{k}+2$ a.e. in $\omega^a_{k,i}$.

(i) $\|g^a_{k,i} - f_k\|_{C^\alpha(\overline{\omega^a_{k,i}})} < 2^{-k-2}\delta$ and $\|(g^a_{k,i})^{-1} - f_k^{-1}\|_{C^\alpha(A^a_{k,i}\overline{\omega^a_{k,i}}+b^a_{k,i})} < 2^{-k-2}\delta$.

(j) $Dg^a_{k,i}(x) \in E_j \cup \bigcup_{b=0}^{n-m} S^b_{j,\tilde{k}+1}$ a.e. in $\omega^a_{k,i}$.

(k) $\left|\{x \in \omega^a_{k,i} : Dg^a_{k,i}(x) \in S^b_{j,\tilde{k}+1}\}\right| \leqslant C(j,\tilde{k},a,b)|\omega^a_{k,i}|$.

In property *(j)* we have used that E_j is open. We define the piecewise affine function

$$f_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in \overline{\omega} \setminus \bigcup_{i=1}^\infty \bigcup_{a=0}^{n-m} \omega^a_{k,i}, \\ g^a_{k,i}(x) & \text{if } x \in \omega^a_{k,i} \text{ for some } i \in \mathbb{N} \text{ and } a \in \{0,\dots,n-m\}, \end{cases}$$

which is a homeomorphism due to Lemma 1.2.2. Property *(a)* holds for $k+1$ since $f_{k+1} = f_k$ on $\partial\omega$. Property *(b)* holds for $k+1$ thanks to *(i)* and Lemma 1.2.2. Property *(f)* for $k+1$ follows easily from the construction. Property *(d)* for $k+1$ follows from *(h)* and *(f)*. Property *(c)* for $k+1$ follows from *(j)* and *(f)*. Finally, we have to prove *(e)* for $k+1$.

By definition of f_{k+1}, we have that, up to a set of measure zero,

$$\omega^b_{k+1} = \bigcup_{a=0}^{n-m} \bigcup_{i=1}^{\infty} \left\{x \in \omega^a_{k,i} : Dg^a_{k,i}(x) \in S^b_{j,\tilde{k}+1}\right\}$$

with disjoint union, hence for $b = 0,\dots,n-m$,

$$\begin{aligned} \frac{|\omega^b_{k+1}|}{|\omega|} &= \sum_{a=0}^{n-m}\sum_{i=1}^{\infty} \frac{|\omega^a_{k,i}|}{|\omega|} \frac{\left|\left\{x \in \omega^a_{k,i} : Dg^a_{k,i}(x) \in S^b_{j,\tilde{k}+1}\right\}\right|}{|\omega^a_{k,i}|} \leqslant \sum_{a=0}^{n-m} \frac{|\omega^a_k|}{|\omega|} C(j,\tilde{k},a,b) \\ &\lesssim \sum_{a=0}^{n-m-1} \tilde{k}^{\frac{1}{2}+a-n} C(j,\tilde{k},a,b) + C(j,\tilde{k},n-m,b)\tilde{k}^{-m} \sum_{d=1}^{k} \tilde{d}^{-\frac{4}{3}}, \end{aligned}$$

where we have used *(k)* and *(e)*. So, in order to prove *(e)* for $k+1$ it is enough to show that there exist $j_0 \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$ and $j \geqslant j_0$ then

$$\sum_{a=0}^{n-m-1} \tilde{k}^{\frac{1}{2}+a-n} C(j,\tilde{k},a,b) + C(j,\tilde{k},n-m,b)\tilde{k}^{-m} \sum_{d=1}^{k} \tilde{d}^{-\frac{4}{3}} \leqslant \left(\tilde{k}+1\right)^{\frac{1}{2}+b-n} \tag{2.2.31}$$

for $0 \leqslant b \leqslant n-m-1$, and

$$\sum_{a=0}^{n-m-1} \tilde{k}^{\frac{1}{2}+a-n} C(j,\tilde{k},a,n-m) + C(j,\tilde{k},n-m,n-m)\tilde{k}^{-m} \sum_{d=1}^{k} \tilde{d}^{-\frac{4}{3}} \leqslant \left(\tilde{k}+1\right)^{-m} \sum_{d=1}^{k+1} \tilde{d}^{-\frac{4}{3}}. \tag{2.2.32}$$

Recall from (2.2.20) that, for $k \geqslant 2$,

$$C(j,\tilde{k},a,b) = (\tilde{k}+1)^{-n+b} \sum_{\ell=\max\{0,a+b-n\}}^{\min\{a,b\}} \binom{a}{\ell}\binom{n-a}{b-\ell} (\tilde{k}+r_{j,\tilde{k}})^{n-a-b+\ell}\, \tilde{k}^{m(\ell-a)+a-b}\, 2^{-j(a-\ell)}.$$

Using the inequality $(\tilde{k}+r_{j,\tilde{k}})^{n-a-b+\ell} \leqslant \tilde{k}^{n-a-b+\ell} + 2^n \tilde{k}^{n-a-b+\ell-1} r_{j,\tilde{k}}$, we find that

$$C(j,\tilde{k},a,b) \leqslant (\tilde{k}+1)^{b-n} \sum_{\ell=\max\{0,a+b-n\}}^{\min\{a,b\}} \binom{a}{\ell}\binom{n-a}{b-\ell} \tilde{k}^{n+m(\ell-a)+\ell-2b} \left[1+2^n r_{j,\tilde{k}} \tilde{k}^{-1}\right] 2^{-j(a-\ell)}.$$

We will also use the cruder inequality

$$C(j,\tilde{k},a,b) \lesssim (\tilde{k}+1)^{b-n}\, \tilde{k}^{n+\min\{a,b\}(m+1)-am-2b}.$$

In order to show inequalities (2.2.31) and (2.2.32), we estimate $C(j,\tilde{k},a,b)$ according to whether a or b equal $n-m$ or are less than it. We first observe that for $a,b \in \{0,\dots,n-m\}$ and $\ell \in \{0,\dots,\min\{a,b\}\}$ we have

$$\begin{cases} m(\ell-a)+\ell+a-2b = 0 & \text{if } a=b=\ell, \\ m(\ell-a)+\ell+a-2b \leqslant -1 & \text{otherwise.} \end{cases}$$

In the case $0 \leqslant b \leqslant n-m-1$ we have that there exists a constant $c_1(n)$ depending on n such that, for $k \geqslant 2$,

$$\begin{aligned} &\left(\tilde{k}+1\right)^{n-b} \sum_{a=0}^{n-m-1} \tilde{k}^{\frac{1}{2}+a-n} C(j,\tilde{k},a,b) \\ &\leqslant \sum_{a=0}^{n-m-1} \sum_{\ell=\max\{0,a+b-n\}}^{\min\{a,b\}} \binom{a}{\ell}\binom{n-a}{b-\ell} \tilde{k}^{m(\ell-a)+\ell+a-2b} \left[\tilde{k}^{\frac{1}{2}} + 2^n r_{j,\tilde{k}} \tilde{k}^{-\frac{1}{2}}\right] 2^{-j(a-\ell)} \\ &\leqslant \tilde{k}^{\frac{1}{2}} + c_1(n) \frac{2^{-j}}{\tilde{k}^{\frac{1}{2}}}, \end{aligned}$$

so

$$\left(\tilde{k}+1\right)^{n-\frac{1}{2}-b} \sum_{a=0}^{n-m-1} \tilde{k}^{\frac{1}{2}+a-n} C(j,\tilde{k},a,b) \leqslant \left(\frac{\tilde{k}}{\tilde{k}+1}\right)^{\frac{1}{2}} + c_1(n) \frac{2^{-j}}{\tilde{k}^{\frac{1}{2}} (\tilde{k}+1)^{\frac{1}{2}}}, \tag{2.2.33}$$

whereas

$$C(j,\tilde{k},n-m,b)\tilde{k}^{-m} \left(\tilde{k}+1\right)^{n-b} \lesssim \tilde{k}^{n+b(m+1)-(n-m)m-2b-m} \leqslant \tilde{k}^{1-m} \leqslant \tilde{k}^{-1}$$

so

$$C(j,\tilde{k},n-m,b)\tilde{k}^{-m} \sum_{d=1}^{k} \tilde{d}^{-\frac{4}{3}} \left(\tilde{k}+1\right)^{n-\frac{1}{2}-b} \leqslant c_1(n) \tilde{k}^{-\frac{3}{2}}. \tag{2.2.34}$$

Given the previous constant $c_1(n)$, let $j_1 \in \mathbb{N}$ be such that for all $j \geqslant j_1$ and $k \geqslant 2$,

$$\left(\frac{\tilde{k}}{\tilde{k}+1}\right)^{\frac{1}{2}} + c_1(n) \left(\frac{2^{-j}}{\tilde{k}^{\frac{1}{2}} (\tilde{k}+1)^{\frac{1}{2}}} + \tilde{k}^{-\frac{3}{2}}\right) \leqslant 1. \tag{2.2.35}$$

Then using (2.2.33), (2.2.34) and (2.2.35)

$$\begin{aligned}&\left(\tilde{k}+1\right)^{n-\frac{1}{2}-b}\left[\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+a-n}C(j,\tilde{k},a,b)+C(j,\tilde{k},n-m,b)\tilde{k}^{-m}\sum_{d=1}^{k}\tilde{d}^{-\frac{4}{3}}\right]\\&\leqslant\left(\frac{\tilde{k}}{\tilde{k}+1}\right)^{\frac{1}{2}}+c_1(n)\left(\frac{2^{-j}}{\tilde{k}^{\frac{1}{2}}(\tilde{k}+1)^{\frac{1}{2}}}+\tilde{k}^{-\frac{3}{2}}\right)\leqslant 1,\end{aligned}$$

which proves (2.2.31). In the case $b=n-m$,

$$\left(\tilde{k}+1\right)^{m}C(j,\tilde{k},a,n-m)\lesssim\tilde{k}^{2m+a-n},$$

so

$$\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+a-n}C(j,\tilde{k},a,n-m)\left(\tilde{k}+1\right)^{m}\lesssim\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+2(m+a-n)}\lesssim\tilde{k}^{-\frac{3}{2}}$$

and, hence, there exists a constant $c_2(n)$ such that

$$\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+a-n}C(j,\tilde{k},a,n-m)\left(\tilde{k}+1\right)^{m}\leqslant c_2(n)\tilde{k}^{-\frac{3}{2}}. \tag{2.2.36}$$

Recall that

$$2^{n}r_{j,\tilde{k}}\leqslant\frac{1}{2}\left(\tilde{k}+1\right)^{-m-1}.$$

Then, splitting the following sum in the case $\ell=n-m$ and the case $\ell<n-m$, we have

$$\begin{aligned}&C(j,\tilde{k},n-m,n-m)\tilde{k}^{-m}(\tilde{k}+1)^{m}\\&\leqslant\sum_{\ell=\max\{0,n-2m\}}^{n-m}\binom{n-m}{\ell}\binom{m}{n-m-\ell}\tilde{k}^{-n+m(\ell-n+m+1)+\ell}\left[1+2^{n}r_{j,\tilde{k}}\tilde{k}^{-1}\right]\\&\leqslant 1+c_2(n)\tilde{k}^{-m-1}.\end{aligned} \tag{2.2.37}$$

In addition, there exists $k_0\in\mathbb{N}$ such that for $k\geqslant k_0$

$$c_2(n)\left(\tilde{k}^{-\frac{3}{2}}+\tilde{k}^{-m-1}\sum_{d=1}^{\infty}\tilde{d}^{-\frac{4}{3}}\right)\leqslant\left(\tilde{k}+1\right)^{-\frac{4}{3}}. \tag{2.2.38}$$

Therefore, for $j\in\mathbb{N}$ and $k\geqslant k_0$ we use (2.2.36), (2.2.37) and (2.2.38) to get

$$\begin{aligned}&(\tilde{k}+1)^{m}\left[\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+a-n}C(j,\tilde{k},a,n-m)+C(j,\tilde{k},n-m,n-m)\tilde{k}^{-m}\sum_{d=1}^{k}\tilde{d}^{-\frac{4}{3}}\right]\\&\leqslant c_2(n)\tilde{k}^{-\frac{3}{2}}+\sum_{d=1}^{k}\tilde{d}^{-\frac{4}{3}}\left(1+c_2(n)\tilde{k}^{-m-1}\right)\leqslant\sum_{d=1}^{k+1}\tilde{d}^{-\frac{4}{3}}.\end{aligned}$$

This proves (2.2.32), which implies *(e)*; thus the construction of $\{f_k\}_{k\in\mathbb{N}}$ is finished.

From *(e)* we obtain

$$\frac{|\omega_k|}{|\omega|}=\sum_{a=0}^{n-m}\frac{|\omega_k^{a}|}{|\omega|}\lesssim\sum_{a=0}^{n-m-1}\tilde{k}^{\frac{1}{2}+a-n}+\tilde{k}^{-m}\sum_{d=1}^{k}\tilde{d}^{-\frac{4}{3}}\lesssim\tilde{k}^{-m}. \tag{2.2.39}$$

By *(b)*, the sequences $\{f_k\}_{k=1}^\infty$ and $\{f_k^{-1}\}_{k=1}^\infty$ converge in the C^α norm. We define f as the limit of f_k. Thanks to the uniform convergence, the limit of f_k^{-1} is the inverse of f. Thus, f is a homeomorphism. In addition, f is piecewise affine. To check this, we see from *(f)* that $f_{k+1} = f_k\chi_{\omega\setminus\omega_k} + g_k\chi_{\omega_k}$ for a certain $g_k : \omega_k \to \mathbb{R}^n$ piecewise affine. Thus, $f_{k+1} = \sum_{i=1}^k g_i\chi_{\omega_i\setminus\omega_{i+1}}$, so $f = \sum_{i=1}^\infty g_i\chi_{\omega_i\setminus\omega_{i+1}}$, which shows that f is piecewise affine. Moreover, $Df_{k+1} = \sum_{i=1}^k Dg_i\chi_{\omega_i\setminus\omega_{i+1}}$, so for any $p \in (1,m)$, thanks to *(h)* and (2.2.39),

$$\begin{aligned}\int_\omega |Df_{k+1}|^p &\leqslant \sum_{i=1}^k (\tilde{i}+2)^p\,(|\omega_i| - |\omega_{i+1}|) \lesssim \sum_{i=1}^k i^p\,(|\omega_i| - |\omega_{i+1}|)\\ &= \sum_{i=1}^k \left[i^p - (i-1)^p\right]|\omega_i| - k^p|\omega_{k+1}| \lesssim \sum_{i=1}^k i^{p-1}|\omega_i| \lesssim \sum_{i=1}^k i^{p-1-m} \lesssim 1,\end{aligned}$$

which shows that $f \in W^{1,p}(\omega,\mathbb{R}^n)$.

Thus, properties *(a)*, *(b)*, *(c)* and (2.2.39) imply properties *i)*, *ii)* and *iii)*. The equalities $f(\omega) = F\omega$ and $f(\overline{\omega}) = \overline{F\omega}$ are a consequence of *i)*.

Finally, we estimate the integrability of Df. Given $t > 0$, let $k_1 = \max\{1, \lfloor t - \mathscr{R}\rfloor\}$. By *(d)* and *(f)*, we obtain that for all $k \geqslant k_1$,

$$|Df_k| \leqslant \tilde{k}_1 + 1 \quad \text{in } \omega\setminus\omega_{k_1},$$

so

$$|Df| \leqslant \tilde{k}_1 + 1 \quad \text{in } \omega\setminus\omega_{k_1}.$$

Therefore,

$$\{x \in \omega : |Df(x)| > t\} \subset \omega_{k_1},$$

hence, from (2.2.39), we obtain

$$\frac{|\{x \in \omega : |Df(x)| > t\}|}{|\omega|} \lesssim \max\{1, t^{-m}\}.$$

Since $\operatorname{dist}(F, \bigcup_{a=0}^{n-m} E^a_{j,|F|}) < r_{j,|F|}$ we have $|F| > \frac{1}{2}$; therefore *iv)* follows. □

Next, we construct a laminate that goes from E_j to E_{j+m}. Again, its proof follows the construction of Subsection 2.2.1.

Lemma 2.2.4. *Let $j \in \mathbb{N}$ and $A \in E_j$. Then there exist $N \in \mathbb{N}\cap[2,2^n]$,*

(2.2.40) $$P_1 \in E_{j+m}; \qquad P_i \in E_{j+m} \cup \bigcup_{a=0}^{n-m} E^a_{j+m}, \quad 2 \leqslant i \leqslant N; \qquad \lambda_i \in [0,1], \quad 1 \leqslant i \leqslant N$$

such that $\nu_A := \sum_{i=1}^N \lambda_i\delta_{P_i}$ belongs to $\mathscr{L}(\mathbb{R}^{n\times n})$, $\overline{\nu_A} = A$, $P_i \neq P_j$ for $i \neq j$,

(2.2.41) $$|A - P_1| < 2^{-j}; \qquad |P_i| = |A|, \quad 1 \leqslant i \leqslant N; \qquad 1 - \lambda_1 \lesssim \frac{2^{-j}}{|A|\max\{1, |A|^{m-1}\}}.$$

Proof. Since $A \in E_j$, there exist $\sigma_1 \leqslant \cdots \leqslant \sigma_n$ and $Q \in SO(n)$ such that $2^{-j-m} < \sigma_i \max\{1, \sigma_n^{m-1}\} < 2^{-j}$ for $1 \leqslant i \leqslant n-m+1$, and $A = Q \operatorname{diag}(\sigma_1, \ldots, \sigma_n) Q^T$.

Since $\rho_{j+m,\sigma_n} = \frac{3 \cdot 2^{-j-2-m}}{\max\{1,\sigma_n^{m-1}\}}$, we have $\rho_{j+m,\sigma_n} < \sigma_1$. As in Lemma 2.2.2, we shall construct families (2.2.23) by finite induction on ℓ. Let $B_{0,0} = A$, $\lambda_{0,0} = 1$ and for $0 \leqslant \ell \leqslant n-1$, $0 \leqslant i \leqslant 2^\ell - 1$, we assume $\{B_{\ell,i}\}_{i=0}^{2^\ell-1}$ and $\{\lambda_{\ell,i}\}_{i=0}^{2^\ell-1}$ have been defined, $Q^T B_{\ell,j} Q$ is diagonal, $\lambda_{\ell,i} \geqslant 0$, equations (2.2.24) and (2.2.25) hold, $|B_{\ell,i}| = \sigma_n$,

$$B_{\ell,0} = Q \operatorname{diag}\Big(\underbrace{\rho_{j+m,\sigma_n}, \ldots, \rho_{j+m,\sigma_n}}_{\min\{\ell,n-m+1\}}, \sigma_{\min\{\ell,n-m+1\}+1}, \ldots, \sigma_n\Big) Q^T,$$

and

$$\lambda_{\ell,0} = \prod_{k=1}^{\min\{\ell,n-m+1\}} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}}.$$

We also assume that if $B_{\ell,i} \notin E_{j+m}$ then

$$\big(Q^T B_{\ell,i} Q\big)_{\alpha,\alpha} \in \{\rho_{j+m,\sigma_n}, \sigma_n\}, \qquad \alpha = 1, \ldots, \ell.$$

With the above induction hypotheses, we construct $\{B_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ and $\{\lambda_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ as follows. For any $0 \leqslant i \leqslant 2^\ell - 1$, let

$$B_{\ell+1,2i} = \begin{cases} B_{\ell,i} - Q \operatorname{diag}\Big(\underbrace{0, \ldots, 0}_{\ell}, \sigma_{\ell+1} - \rho_{j+m,\sigma_n}, \underbrace{0, \ldots, 0}_{n-\ell-1}\Big) Q^T & \text{if } B_{\ell,i} \notin E_{j+m}, \\ B_{\ell,i} & \text{if } B_{\ell,i} \in E_{j+m}, \end{cases}$$

$$B_{\ell+1,2i+1} = \begin{cases} B_{\ell,i} - Q \operatorname{diag}\Big(\underbrace{0, \ldots, 0}_{\ell}, \sigma_{\ell+1} - \sigma_n, \underbrace{0, \ldots, 0}_{n-\ell-1}\Big) Q^T & \text{if } B_{\ell,i} \notin E_{j+m}, \\ B_{\ell,i} & \text{if } B_{\ell,i} \in E_{j+m}, \end{cases}$$

$$\lambda_{\ell+1,2i} = \begin{cases} \frac{\sigma_n - \sigma_{\ell+1}}{\sigma_n - \rho_{j+m,\sigma_n}} \lambda_{\ell,i} & \text{if } B_{\ell,i} \notin E_{j+m}, \\ \lambda_{\ell,i} & \text{if } B_{\ell,i} \in E_{j+m}, \end{cases}$$

$$\lambda_{\ell+1,2i+1} = \begin{cases} \frac{\sigma_{\ell+1} - \rho_{j+m,\sigma_n}}{\sigma_n - \rho_{j+m,\sigma_n}} \lambda_{\ell,i} & \text{if } B_{\ell,i} \notin E_{j+m}, \\ 0 & \text{if } B_{\ell,i} \in E_{j+m}. \end{cases}$$

Hence

$$B_{\ell,i} = \frac{\sigma_n - \sigma_{\ell+1}}{\sigma_n - \rho_{j+m,\sigma_n}} B_{\ell+1,2i} + \frac{\sigma_{\ell+1} - \rho_{j+m,\sigma_n}}{\sigma_n - \rho_{j+m,\sigma_n}} B_{\ell+1,2i+1}.$$

Using that $2^{-j-2m} < \rho_{j+m,\sigma_n} \sigma_n < 2^{-j-m}$ and the definition of E_j, we have that $B_{\ell,0} \in E_{j+m}$ if and only if $\ell \geqslant n-m+1$. Hence, if $B_{\ell,0} \in E_{j+m}$,

$$\lambda_{\ell+1,0} = \lambda_{\ell,0} = \prod_{k=1}^{n-m+1} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}} = \prod_{k=1}^{\min\{\ell+1,n-m+1\}} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}},$$

whereas if $B_{\ell,0} \notin E_{j+m}$,

$$\lambda_{\ell+1,0} = \frac{\sigma_n - \sigma_{\ell+1}}{\sigma_n - \rho_{j+m,\sigma_n}} \lambda_{\ell,i} = \prod_{k=1}^{\ell+1} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}} = \prod_{k=1}^{\min\{\ell+1,n-m+1\}} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}}.$$

With this, it is clear that $\{B_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ and $\{\lambda_{\ell+1,i}\}_{i=0}^{2^{\ell+1}-1}$ satisfy the induction hypotheses.

For $i = 1,\dots,2^n$ define $\hat{\lambda}_i = \lambda_{n,i-1}$ and $\hat{P}_i = B_{n,i-1}$. We can assume that the $\hat{P}_i$'s are distinct; otherwise, we put together the copies and add the corresponding coefficients. Hence, we get $N \in \mathbb{N} \cap [2,2^n]$, $P_1,\dots,P_N$ all different, and $\lambda_1,\dots,\lambda_N \in [0,1]$ as in the statement.

We have shown that $\nu_A \in \mathscr{L}(\mathbb{R}^{n\times n})$ and $\overline{\nu_A} = B_{0,0} = A$. Now we estimate the distance between A and P_1:

$$\begin{aligned} |A - P_1| &= |Q \operatorname{diag}\Big(\underbrace{\sigma_1 - \rho_{j+m,\sigma_n},\dots,\sigma_{n-m+1} - \rho_{j+m,\sigma_n}}_{n-m+1}, \underbrace{0,\dots,0}_{m-1}\Big) Q^T| \\ &= \sigma_{n-m+1} - \rho_{j+m,\sigma_n} < \sigma_{n-m+1} < 2^{-j}. \end{aligned}$$

To finish the proof it only remains to check the last estimate of (2.2.41). Notice that

$$\lambda_1 \geqslant \hat{\lambda}_1 = \prod_{k=1}^{n-m+1} \frac{\sigma_n - \sigma_k}{\sigma_n - \rho_{j+m,\sigma_n}} = \prod_{k=1}^{n-m+1} \left(1 - \frac{\sigma_k - \rho_{j+m,\sigma_n}}{\sigma_n - \rho_{j+m,\sigma_n}}\right) \geqslant \prod_{k=1}^{n-m+1} \left(1 - \frac{\sigma_k}{\sigma_n}\right).$$

If $|A| \geqslant 1$, for $1 \leqslant k \leqslant n-m+1$ we have $\sigma_k \sigma_n^{m-1} < 2^{-j}$, so

$$1 - \lambda_1 \leqslant 1 - \prod_{k=1}^{n-m+1} \left(1 - \frac{2^{-j}}{\sigma_n^m}\right) = 1 - \left(1 - \frac{2^{-j}}{\sigma_n^m}\right)^{n-m+1} \lesssim \frac{2^{-j}}{\sigma_n^m} = \frac{2^{-j}}{|A|^m}.$$

If, on the other hand, $|A| < 1$, then $\sigma_k < 2^{-j}$ for $1 \leqslant k \leqslant n-m+1$, and since $A \in E_j$, we know that $|A| > \frac{1}{2}$, so

$$1 - \lambda_1 \leqslant 1 - \prod_{k=1}^{n-m+1} \left(1 - \frac{2^{-j}}{\sigma_n}\right) = 1 - \left(1 - \frac{2^{-j}}{\sigma_n}\right)^{n-m+1} \lesssim \frac{2^{-j}}{\sigma_n} = \frac{2^{-j}}{|A|}$$

and the proof is finished. □

A variant of Lemma 2.2.4 will be needed. If, instead of starting from an $A \in E_j$, we begin with the identity matrix, the same proof of Lemma 2.2.4 yields the following result, which will be used in the first step of the construction of the sequence approximating the final homeomorphism of Theorem 2.2.1.

Lemma 2.2.5. *Given $\alpha \in (0,1)$ and $\delta > 0$, let $j_1 \in \mathbb{N}$ be as in Lemma 2.2.3. Then there exist $N \in \mathbb{N} \cap [2,2^n]$,*

$$P_1 \in E_{j_1}; \qquad P_i \in E_{j_1} \cup \bigcup_{a=0}^{n-m} E_{j_1}^a, \quad 2 \leqslant i \leqslant N; \qquad \lambda_i \in [0,1], \quad 1 \leqslant i \leqslant N$$

such that $\nu_I := \sum_{i=1}^N \lambda_i \delta_{P_i}$ belongs to $\mathscr{L}(\mathbb{R}^{n\times n})$, $\overline{\nu_I} = I$, $P_i \neq P_j$ for $i \neq j$ and

$$|I| \lesssim |P_i| \lesssim |I|, \qquad 1 \leqslant i \leqslant N.$$

Next, we use the laminate of Lemma 2.2.4 to construct a homeomorphism with the properties mentioned in Lemma 2.2.6 below.

Lemma 2.2.6. *Let $A \in E_j$. For any bounded open $\omega \subset \mathbb{R}^n$, $\alpha \in (0,1)$ and $\eta > 0$ there exists a piecewise affine homeomorphism $h \in W^{1,1}(\omega,\mathbb{R}^n) \cap C^\alpha(\overline{\omega})$ satisfying*

(a) $h(x) = Ax$ *on* $\partial\omega$.

(b) $\|h - A\|_{C^\alpha(\overline{\omega})} < \eta$ *and* $\|h^{-1} - A^{-1}\|_{C^\alpha(\overline{A\omega})} < \eta$.

(c) $Dh(x) \in E_{j+m}$ *a.e.* $x \in \omega$.

(d) $\int_\omega |Dh(x) - A|dx \lesssim 2^{-j}|\omega|$.

(e) There exists an open set $\tilde{\omega} \subset \omega$ *such that*

- *(e1)* $|Dh(x) - A| \lesssim 2^{-j}$ *a.e.* $x \in \omega \setminus \tilde{\omega}$.
- *(e2)* $|\{x \in \tilde{\omega} : |Dh(x)| > t\}| \lesssim 2^{-j}|\omega|t^{-m}$.

Proof. First we build the laminate of Lemma 2.2.4:

$$\nu_A = \sum_{i=1}^{N} \lambda_i \delta_{P_i} \in \mathscr{L}(\mathbb{R}^{n\times n})$$

satisfying $\overline{\nu_A} = A$, (2.2.40) and (2.2.41). Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\left\{\frac{1}{2} r_{j,|A|},\ 2^{-j} - |A - P_1|,\ \frac{1}{2}\min_{2\leqslant i\leqslant N} |P_1 - P_i|\right\}$$

and

$$r_{j,|A|} < 2r_{j,\mathscr{R}} \qquad \text{for } \mathscr{R} \in (|A| - \varepsilon, |A| + \varepsilon).$$

Then, Proposition 1.3.5 gives a piecewise affine homeomorphism $g : \omega \to A\omega$ satisfying

1) $g(x) = Ax$ on $\partial\omega$,

2) $\|g - A\|_{C^\alpha(\overline{\omega})} < \frac{\eta}{2}$ and $\|g^{-1} - A^{-1}\|_{C^\alpha(A\overline{\omega})} < \frac{\eta}{2}$,

3) $|\{x \in \omega : |Dg(x) - P_i| < \varepsilon\}| = \lambda_i|\omega|$ for $i = 1,\dots,N$.

Let $\tilde{\omega} = \{x \in \omega : g$ is affine in a neighbourhood of x and $|Dg(x) - P_1| > \varepsilon\}$ and

$$\hat{\omega} = \left\{ x \in \omega : \begin{array}{l} g \text{ is affine in a neighbourhood of } x \text{ and} \\ |Dg(x) - P_i| < \varepsilon \text{ for some } i \in \{2,\dots,N\} \text{ and } P_i \in \bigcup_{a=0}^{n-m} E^a_{j+m} \end{array} \right\}.$$

As $\operatorname{dist}(E_{j+m}, \bigcup_{a=0}^{n-m} E^a_{j+m}) > \frac{1}{2}$ and $\varepsilon < \frac{1}{4}$, we have that $\hat{\omega} \subset \tilde{\omega}$. Note also that $\hat{\omega}$ and $\tilde{\omega}$ are open. Finally, the choice of ε was done so that, thanks to 3) the set of $x \in \Omega$ such that $|Dg(x) - P_1| = \varepsilon$ has measure zero.

Since g is piecewise affine, there exist a family $\{\hat{\omega}_k\}_{k\in\mathbb{N}}$ of open sets such that $\hat{\omega} = \bigcup_{k=1}^{\infty} \hat{\omega}_k$, and $\hat{P}_k \in \mathbb{R}^{n\times n}$, $b_k \in \mathbb{R}^n$ with $g(x) = \hat{P}_k x + b_k$ in $\hat{\omega}_k$. Recalling that $|P_i| = |A|$ for $i = 1,\dots,N$, and $||A| - |\hat{P}_k|| < \varepsilon$, we have

$$\operatorname{dist}\left(\hat{P}_k, \bigcup_{a=0}^{n-m} E^a_{j+m}\right) < \varepsilon < \frac{1}{2} r_{j,|A|} < r_{j,|\hat{P}_k|}, \qquad k \in \mathbb{N}.$$

We define h as the piecewise affine homeomorphism given by Lemma 2.2.3 in each $\hat{\omega}_k$ and as g in $\overline{\omega} \setminus \bigcup_{k=1}^{\infty} \hat{\omega}_k$. By Lemma 1.2.2, h is a homeomorphism, and satisfies *(a)*, *(b)* and *(e1)*. Property *(c)* comes from *(iii)* in Lemma 2.2.3. By *(iv)* of the same lemma we have

$$\frac{|\{x \in \hat{\omega}_k : |Dh(x)| > t\}|}{|\hat{\omega}_k|} \lesssim |\hat{P}_k|^m t^{-m}, \qquad t > 0.$$

Therefore, using (2.2.41) and that for all k there exists i such that $|\hat{P}_k - P_i| < \varepsilon$, we have

$$\frac{|\{x \in \hat{\omega} : |Dh(x)| > t\}|}{|\hat{\omega}|} \lesssim |A|^m t^{-m}.$$

Now, by (2.2.41) and 3), we have

$$||Dh| - |A|| \leqslant |Dh - P_i| = \left|Dg - P_i\right| < \varepsilon \qquad \text{a.e. in } \omega \setminus \hat{\omega},$$

for some $i \in \{1,\dots,N\}$ depending on the point. Hence $|Dh| < |A| + \varepsilon$ a.e. in $\tilde{\omega} \setminus \hat{\omega}$. Thus, if $t > |A| + \varepsilon$ we have

$$|\{x \in \tilde{\omega} : |Dh(x)| > t\}| = |\{x \in \hat{\omega} : |Dh(x)| > t\}| \lesssim |A|^m t^{-m} |\hat{\omega}| \leqslant |A|^m t^{-m} |\tilde{\omega}|,$$

whereas, if $0 < t \leqslant |A| + \varepsilon$ we get $1 \lesssim |A|^m t^{-m}$ and

$$\begin{aligned} |\{x \in \tilde{\omega} : |Dh(x)| > t\}| &= |\{x \in \tilde{\omega} \setminus \hat{\omega} : |Dh(x)| > t\}| + |\{x \in \hat{\omega} : |Dh(x)| > t\}| \\ &\lesssim |\tilde{\omega} \setminus \hat{\omega}| + |A|^m t^{-m} |\hat{\omega}| \lesssim |A|^m t^{-m} |\tilde{\omega}|. \end{aligned}$$

Hence, for all $t > 0$,

$$|\{x \in \tilde{\omega} : |Dh(x)| > t\}| \lesssim |A|^m t^{-m} |\tilde{\omega}|. \tag{2.2.42}$$

So using (2.2.42) and

$$|\tilde{\omega}| = (1 - \lambda_1)|\omega| \lesssim \frac{2^{-j}}{|A| \max\{1, |A|^{m-1}\}} |\omega|, \tag{2.2.43}$$

we have $|\{x \in \tilde{\omega} : |Dh(x)| > t\}| \lesssim 2^{-j} t^{-m} |\omega|$, that is, *(e2)*.

Finally,

$$\int_\omega |Dh(x) - A|\, dx \leqslant \int_{\tilde{\omega}} |Dh(x)|\, dx + |A|\, |\tilde{\omega}| + \int_{\omega \setminus \tilde{\omega}} |Dh(x) - A|\, dx. \tag{2.2.44}$$

Using now the common formula for calculating the L^1 norm of a function in terms of its distribution function, as well as (2.2.42), we obtain

$$\int_{\tilde{\omega}} |Dh(x)|\,dx = \int_0^{2|A|} |\{x \in \tilde{\omega} : |Dh(x)| > t\}|\,dt + \int_{2|A|}^{\infty} |\{x \in \tilde{\omega} : |Dh(x)| > t\}|\,dt$$
$$\lesssim |A|\,|\tilde{\omega}| + |A|^m\,|\tilde{\omega}| \int_{2|A|}^{\infty} t^{-m}\,dt \lesssim |A|\,|\tilde{\omega}|,$$

hence, thanks to (2.2.43),

$$\int_{\tilde{\omega}} |Dh(x)|\,dx + |A|\,|\tilde{\omega}| \lesssim |A|\,|\tilde{\omega}| \leqslant 2^{-j}\,|\omega|, \tag{2.2.45}$$

while *(e1)* yields

$$\int_{\omega\setminus\tilde{\omega}} |Dh(x) - A|\,dx \lesssim 2^{-j}\,|\omega \setminus \tilde{\omega}| \leqslant 2^{-j}\,|\omega|. \tag{2.2.46}$$

Inequalities (2.2.44), (2.2.45) and (2.2.46) show *(d)* and finish the proof. □

The next lemma is the analogous of the previous one when one starts with the identity matrix. Its proof is similar to that of Lemma 2.2.6, but using Lemma 2.2.5 instead of Lemma 2.2.4.

Lemma 2.2.7. *For any $\alpha \in (0,1)$ and $\eta > 0$ there exist $j_1 \in \mathbb{N}$ and a piecewise affine homeomorphism $h \in W^{1,1}(\Omega,\mathbb{R}^n) \cap C^{\alpha}(\overline{\Omega},\mathbb{R}^n)$ satisfying*

(a) $h(x) = x$ *on* $\partial\Omega$.

(b) $\|h - I\|_{C^{\alpha}(\overline{\Omega})} < \eta$ *and* $\|h^{-1} - I\|_{C^{\alpha}(\overline{\Omega})} < \eta$.

(c) $Dh(x) \in E_{j_1}$ *a.e.* $x \in \Omega$.

(d) $\int_{\Omega} |Dh(x) - I|\,dx \lesssim \frac{1}{2}$.

(e) $|\{x \in \Omega : |Dh(x)| > t\}| \lesssim |\Omega|\,t^{-m}$ *for all* $t > 0$.

2.2.3 Proof of Theorem 2.2.1

We are in a position to prove Theorem 2.2.1 using Lemmas 2.2.6 and 2.2.7.

Proof of Theorem 2.2.1. Let $j_1 \in \mathbb{N}$ be as in Lemma 2.2.3. For each $j \in \mathbb{N}$ we will construct a piecewise affine homeomorphism $f_j \in W^{1,1}(\Omega,\mathbb{R}^n) \cap C^{\alpha}(\overline{\Omega},\mathbb{R}^n)$ such that

1) $f_j(x) = x$ on $\partial\Omega$.

2) $\|f_j - f_{j-1}\|_{C^{\alpha}(\overline{\Omega})} < 2^{-j}\delta$ and $\|f_j^{-1} - f_{j-1}^{-1}\|_{C^{\alpha}(\overline{\Omega})} < 2^{-j}\delta$.

3) $Df_j \in E_{j_1+(j-1)m}$ a.e.

4) $\int_{\Omega} |Df_j(x) - Df_{j-1}(x)| dx \lesssim 2^{-j}\,|\Omega|$.

5) $|\{x \in \Omega : |Df_j(x)| > t\}| \lesssim |\Omega|\, t^{-m} \prod_{k=0}^{j-1}(1-2^{-k-1})^{-m}(1+2^{-k})$.

The construction of f_j proceeds by induction. Let $f_0 = \mathrm{id}$, which does not satisfy 3). We use Lemma 2.2.7 to create a piecewise affine homeomorphism f_1 such that properties 1–5) hold for $j = 1$.

Now suppose we have f_j. Since f_j is piecewise affine, for each $i \in \mathbb{N}$ there exist $A_{ij} \in \mathbb{R}^{n\times n}$, $b_{ij} \in \mathbb{R}^n$ and $\Omega_{ij} \subset \Omega$ open such that $|\Omega \setminus \bigcup_i \Omega_{ij}| = 0$ and

$$f_j(x) = A_{ij}x + b_{ij}, \quad x \in \Omega_{ij}$$

and, by 3), $A_{ij} \in E_{j_1+(j-1)m}$. On each Ω_{ij} we apply Lemma 2.2.6: there exists a piecewise affine homeomorphism $h_{ij} : \overline{\Omega}_{ij} \to A_{ij}\overline{\Omega}_{ij} + b_{ij}$ in $W^{1,1}$ and in C^α such that

- $h_{ij}(x) = f_j(x)$ for $x \in \partial\Omega_{ij}$.
- $\|h_{ij} - f_j\|_{C^\alpha(\overline{\Omega_{ij}})} < 2^{-(j+2)}\delta$ and $\|h_{ij}^{-1} - f_j^{-1}\|_{C^\alpha(\overline{\Omega}_{ij}+b_{ij})} < 2^{-(j+2)}\delta$.
- $Dh_{ij}(x) \in E_{j_1+jm}$ a.e. $x \in \Omega_{ij}$.
- $\int_{\Omega_{ij}} |Dh_{ij}(x) - A_{ij}|\,dx \lesssim 2^{-j}|\Omega_{ij}|$.
- There exists an open set $\tilde{\Omega}_{ij} \subset \Omega_{ij}$ such that

$$|Dh_{ij}(x) - A_{ij}| \lesssim 2^{-j} \text{ a.e. } x \in \Omega_{ij} \setminus \tilde{\Omega}_{ij} \quad \text{and} \quad \frac{\left|\{x \in \tilde{\Omega}_{ij} : |Dh_{ij}(x)| > t\}\right|}{\left|\Omega_{ij}\right|} \lesssim 2^{-j}\, t^{-m}.$$

We define the piecewise affine function $f_{j+1} : \overline{\Omega} \to \mathbb{R}^n$ as

$$f_{j+1} := \begin{cases} f_j(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} \Omega_{ij}, \\ h_{ij}(x) & \text{if } x \in \Omega_{ij} \text{ for some } i \in \mathbb{N}. \end{cases}$$

By Lemma 1.2.2, it is a homeomorphism and, moreover, properties 1)–4) hold for $j+1$. In addition,

(a) $\exists C > 0$ depending only on n such that $|Df_{j+1} - Df_j| \leqslant C2^{-j}$ a.e. in $\bigcup_i \left(\Omega_{ij} \setminus \tilde{\Omega}_{ij}\right)$.

(b) $\left|\{x \in \bigcup_{i\in\mathbb{N}} \tilde{\Omega}_i : |Df_{j+1}(x)| > t\}\right| \lesssim 2^{-j}|\Omega|t^{-m}$.

To get property 5) for f_{j+1} we proceed as follows. Let

$$t > \frac{1+2^{-1}C}{1+2^{-2}}.$$

On the one hand, thanks to (a) and 5),

$$\begin{aligned}\sum_{i=1}^{\infty} \left|\{x \in \Omega_{ij} \setminus \tilde{\Omega}_{ij} : |Df_{j+1}(x)| > t\}\right| &\leqslant \left|\{x \in \Omega : |Df_j(x)| > t - C2^{-j}\}\right| \\ &\lesssim |\Omega|\,(t - C2^{-j})^{-m} \prod_{k=0}^{j-1}(1-2^{-k-1})^{-m}(1+2^{-k}),\end{aligned}$$

but $(t-C2^{-j})^{-m} \leqslant (1-2^{-j-1})^{-m}t^{-m}$, hence

$$\sum_{i=1}^{\infty}\left|\{x\in\Omega_{ij}\setminus\tilde{\Omega}_{ij}:|Df_{j+1}(x)|>t\}\right| \lesssim |\Omega|\,(1-2^{-j-1})^{-m}t^{-m}\prod_{k=0}^{j-1}(1-2^{-k-1})^{-m}(1+2^{-k}).$$

Summing this estimate with that of (b) we obtain

$$\begin{aligned}\left|\{x\in\Omega:|Df_{j+1}(x)|>t\}\right| &\lesssim |\Omega|\,t^{-m}\left[(1-2^{-j-1})^{-m}\prod_{k=0}^{j-1}(1-2^{-k-1})^{-m}(1+2^{-k})+2^{-j}\right]\\ &\leqslant |\Omega|\,t^{-m}\prod_{k=0}^{j}(1-2^{-k-1})^{-m}(1+2^{-k}),\end{aligned}$$

so property 5) holds for $j+1$. This concludes the construction of $\{f_j\}_{j\in\mathbb{N}}$ with properties 1)–5).

As in the proof of Lemma 2.2.3, property 2) implies that the sequence $\{f_j\}_{j=1}^{\infty}$ converges in C^{α} to a function f that is a homeomorphism with a C^{α} inverse. It also shows property *v)* of the statement. Property 4), on the other hand, shows that $\{Df_j\}_{j\in\mathbb{N}}$ converges in L^1. Consequently, $f\in W^{1,1}(\Omega,\mathbb{R}^n)$.

Now, for a subsequence $Df_j\to Df$ a.e., so, thanks to the continuity of the singular values (see, e.g., (2.2.21)), we obtain from property 3) that $Df(x)\in\Gamma_+$ and $\operatorname{rank}\big(Df(x)\big)<m$ a.e. in Ω. From the convergence $Df_j\to Df$ in measure and property 5) we have

$$\frac{|\{x\in\Omega:|Df(x)|>t\}|}{|\Omega|}\lesssim t^{-m}\prod_{k=0}^{\infty}(1-2^{-k-1})^{-m}(1+2^{-k})\lesssim t^{-m},$$

and therefore, $Df\in L^{m,w}(\Omega)$.

Finally, we have to prove that f is the gradient of a strictly convex function. Applying Lemma 1.3.1 we have that f is the gradient of a convex function $u\in W^{2,1}(\Omega)$.

To see that u is strictly convex we follow the same reasoning appearing in [106]. Without loss of generality we can suppose that $\operatorname{diam}(\Omega_{i,j})<2^{-j}$ for all $i,j\in\mathbb{N}$. Hence, given $x,y\in\Omega$, $x\neq y$, let j_0 be such that $2^{-j_0+1}<|x-y|$. Then, there exist $i_x,i_y\in\mathbb{N}$ such that $x\in\overline{\Omega_{i_x,j_0}}$, $y\in\overline{\Omega_{i_y,j_0}}$ and $\operatorname{dist}(\overline{\Omega_{i_x,j_0}},\overline{\Omega_{i_y,j_0}})>0$. Denote by $[x,y]$ the segment between x and y, and let $x_0:=x+a(y-x)\in\partial\Omega_{i_x,j_0}\cap[x,y]$ and $y_0:=x+b(y-x)\in\partial\Omega_{i_y,j_0}\cap[x,y]$ for $0\leqslant a<b\leqslant 1$. On the other hand, using that for all $j\geqslant j_0$ we have $f_j|_{\partial\Omega_{i_x,j_0}}=f_{j_0}|_{\partial\Omega_{i_x,j_0}}$ and $f_j|_{\partial\Omega_{i_y,j_0}}=f_{j_0}|_{\partial\Omega_{i_y,j_0}}$ we get, by the uniform convergence of f_j, that $\nabla u(x_0)=f(x_0)=f_{j_0}(x_0)$ and $\nabla u(y_0)=f(y_0)=f_{j_0}(y_0)$. Since Df_{j_0} is positive definite a.e. we get that $f_{j_0}=\nabla u_{j_0}$ with u_{j_0} being strictly convex. Using that the function

$$t\to u_{j_0}(x+t(y-x))$$

is strictly convex on $[0,1]$, we have

$$\nabla u(x_0)(y-x)=\nabla u_{j_0}(x_0)(y-x)<\nabla u_{j_0}(y_0)(y-x)=\nabla u(y_0)(y-x).$$

So, thanks to the convexity of u,

$$\nabla u(x)(y-x)\leqslant\nabla u(x_0)(y-x)<\nabla u(y_0)(y-x)\leqslant\nabla u(y)(y-x).$$

Therefore, u is strictly convex. Theorem 2.2.1 is proved. □

In fact, since in our construction the sets E_j approximate planes of dimension $m-1$, our function satisfies $\operatorname{rank}\big(Df(x)\big)=m-1$ a.e. in Ω.

2.3 Bi-Sobolev homeomorphisms f with Df and Df^{-1} of low rank

This section is devoted to the proof of the following theorem in which we construct a bi-Sobolev homeomorphism f with Df and Df^{-1} of low rank. The integrabilities are optimal in view of Theorem 2.1.5.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $m_1, m_2 \in \mathbb{N}$, $1 \leqslant m_1, m_2 \leqslant n-2$, $\varepsilon > 0$ and $\alpha \in (0,1)$. Then there exists a strictly convex function $u : \Omega \to \mathbb{R}$, $u \in W^{2,1}(\Omega)$, whose gradient $f = \nabla u : \Omega \to \Omega$ is a bi-Sobolev homeomorphism and satisfies:*

i) $f = \mathrm{id}$ *on* $\partial\Omega$.

ii) $\|f - \mathrm{id}\|_{C^\alpha(\overline{\Omega})} < \varepsilon$ *and* $\|f^{-1} - \mathrm{id}\|_{C^\alpha(\overline{\Omega})} < \varepsilon$.

iii) $\mathrm{rank}(Df(x)) = m_1$ *a.e.* $x \in \Omega$ *and* $\mathrm{rank}(Df^{-1}(y)) = m_2$ *for a.e.* $y \in \Omega$.

iv) $Df \in \bigcap_{p<m_1'+1} L^p(\Omega, \mathbb{R}^{n\times n})$ *and* $Df^{-1} \in \bigcap_{q<m_2'+1} L^q(\Omega, \mathbb{R}^{n\times n})$,

where for $i = 1,2$ we have

$$m_i' = \begin{cases} m_i & \text{if } m_1 + m_2 \leqslant n-1, \\ m_i - (m_1 + m_2 - n + 1) & \text{if } m_1 + m_2 \geqslant n. \end{cases}$$

2.3.1 Construction of the laminate in dimension three

In this section we construct the sequence of laminates ν_j of finite order that is behind the proof of Theorem 2.3.1 in the case $n = 3$ and $m_1 = m_2 = 1$. The actual proof will consist in approximating ν_k with laminates of finite order supported in the set of positive definite matrices, then use Proposition 1.3.5 to obtain homeomorphisms that are close to the approximate laminates and such that their inverse are close to the inverse of that laminates, then paste the obtained homeomorphisms to construct a homeomorphism in the whole domain and, finally, a limit passage will yield the homeomorphism f of Theorem 2.3.1.

Although this section is not strictly necessary for the proof of Theorem 2.3.1, it will help the reader to follow the construction of Subsection 2.3.2.

To define ν_j we need to define the following sets. For $i, k \in \mathbb{N}$ let

$$A_k^i = \left\{A \in \Gamma_+ : \sigma_j(A) = k^{-1} \text{ for } j \in \{1,2\} \text{ and } \sigma_3(A) \in \{i-1, i\} \setminus \{0\}\right\}, \tag{2.3.1}$$

$$B_k^i = \left\{A \in \Gamma_+ : \sigma_1(A) \in \{k^{-1}, (k-1)^{-1}\} \text{ and } \sigma_j(A) = i \text{ for } j \in \{2,3\}\right\} \setminus \{I\}. \tag{2.3.2}$$

Observe that a matrix $A \in \Gamma_+ \setminus \{I\}$ belongs to A_k^i if and only if A^{-1} is in B_i^k.

The laminates ν_j to be constructed will satisfy the following:

(a) $\overline{\nu}_j = I$,

(b) $\mathrm{supp}(\nu_j) \subset \bigcup_{i=1}^j A_j^i \cup B_i^j$.

(c) For all $\varepsilon > 0$ there exists a bounded family of constants $\{C_j\}_{j\in\mathbb{N}}$, such that, for all $j \in \mathbb{N}$,

$$\nu_j(A^i_j) \leqslant C_j i^{-3+\varepsilon} \text{ and } \nu_j(B^j_i) \leqslant C_j i^{-2+\varepsilon} j^{-2}.$$

When we approximate these laminates by functions, f_j, and then pass to the limit, we obtain a bi-Sobolev homeomorphism f that is the identity on the border due to (a); by (b) and (c) we get for f_j

$$Df_j \in \bigcup_{i=1}^{j} \left(A^i_j \cup B^j_i\right) + B(0, r_j) \text{ with } r_j \to 0,$$

$$|\{x : Df_j(x) \in \bigcup_{i=1}^{j} B^j_i + B(0, r_j)\}| \to 0,$$

and, for the inverse, using that $A \in A^i_j$ if and only if $A^{-1} \in B^j_i$, we obtain

$$Df_j^{-1} \in \bigcup_{i=1}^{j} \left(A^i_j \cup B^j_i\right) + B(0, r'_j) \text{ with } r'_j \to 0,$$

$$|\{y : Df_j^{-1}(y) \in \bigcup_{i=1}^{j} B^j_i + B(0, r'_j)\}| \to 0.$$

So, the ranks of Df and Df^{-1} are equal to 1 almost everywhere. Moreover, with (c) we obtain the following. Let $t, \varepsilon > 0$, and pick $j \in \mathbb{N}$, $j > t$ and big enough; then

$$\nu_j\left(\{A \in \mathbb{R}^{3\times 3} : |A| > t\}\right) \lesssim \sum_{i=\lceil t\rceil}^{j} i^{-3+\varepsilon} + j^{-2} \sum_{i=1}^{j} i^{-2+\varepsilon} \lesssim t^{-2+\varepsilon},$$

$$\begin{aligned}\nu_j^{-1}\left(\{A \in \mathbb{R}^{3\times 3} : |A| > t\}\right) &\lesssim \sum_{i=1}^{j} i^{-3+\varepsilon} \sup_{M\in A^i_j} \{\det M\} + j^{-2} \sum_{i=\lceil t\rceil}^{j} i^{-2+\varepsilon} \sup_{M\in B^j_i} \{\det M\} \\ &\leqslant j^{-2} \sum_{i=1}^{j} i^{-2+\varepsilon} + 2 \sum_{i=\lceil t\rceil}^{j} i^{-3+\varepsilon} \lesssim t^{-2+\varepsilon},\end{aligned}$$

where ν_j^{-1} is the inverse laminate, according to Definition 1.0.4 Therefore, this gives us that $Df, Df^{-1} \in \bigcap_{p<2} W^{1,p}$, respectively.

The laminates ν_j are defined inductively as follows. We begin with $\nu_1 = \delta_I$ (where I is the identity matrix). It is clear that ν_1 satisfies (a), (b) and (c). Now, let

$$\nu_j = \sum_{k=1}^{N} \lambda_k \delta_{A_k} \in \mathscr{L}\left(\mathbb{R}^{n\times n}\right), \tag{2.3.3}$$

with $A_k \in \bigcup_{i=1}^{j}(A^i_j \cup B^j_i)$, all different. For each $A \in \operatorname{supp}(\nu_j)$ we are going to construct a laminate ν_A, whose support is in $\bigcup_{i=1}^{j+1}(A^i_{j+1} \cup B^{j+1}_i)$. To do that, we need the following two lemmas.

Lemma 2.3.2. *Let $A \in A_k^i$. Then there exists a laminate of finite order ν such that*

- $\overline{\nu} = A$,
- $\operatorname{supp}(\nu) \subset A_{k+1}^i \cup B_{k+1}^i$,
- $\nu(B_{k+1}^i) \lesssim \left(k^2 i\right)^{-1}$.

Lemma 2.3.3. *Let $A \in B_k^i$. Then there exists a laminate of finite order ν such that*

- $\overline{\nu} = A$,
- $\operatorname{supp}(\nu) \subset A_k^{i+1} \cup B_k^{i+1}$,
- $\nu(A_k^{i+1}) \lesssim i^{-1}$,
- $\nu(B_k^{i+1}) - \left(\frac{i}{i+1}\right)^2 \lesssim (ik)^{-2}$.

We will only prove Lemma 2.3.2, the proof of Lemma 2.3.3 being analogous.

Proof of Lemma 2.3.2. Given $A \in A_k^i$, without loss of generality, we can assume that

$$A = \operatorname{diag}(k^{-1}, k^{-1}, \sigma), \text{ with } \sigma \in \{\max\{i-1, 1\}, i\}.$$

Now, we denote $\mu = \frac{i-k^{-1}}{i-(k+1)^{-1}}$; observe that μ satisfies

$$0 < \mu \leqslant 1 \qquad \text{and } 1-\mu \leqslant (k^2 i)^{-1}. \tag{2.3.4}$$

Using that

$$k^{-1} = \mu(k+1)^{-1} + (1-\mu)i, \tag{2.3.5}$$

we split A in the following way:

$$\begin{tikzcd}[arrows=dash]
 & & ((k+1)^{-1},(k+1)^{-1},\sigma) \in A_{k+1}^i \\
 & ((k+1)^{-1},k^{-1},\sigma) \arrow[ur, "\mu"] \arrow[dr, "1-\mu"'] & \\
(k^{-1},k^{-1},\sigma) \arrow[ur, "\mu"] \arrow[dr, "1-\mu"'] & & ((k+1)^{-1},i,\sigma) \\
 & (i,k^{-1},\sigma). &
\end{tikzcd}$$

Therefore

$$A = \mu^2 \operatorname{diag}((k+1)^{-1},(k+1)^{-1},\sigma) + \mu(1-\mu)\operatorname{diag}((k+1)^{-1},i,\sigma) + (1-\mu)\operatorname{diag}(i,k^{-1},\sigma).$$

If $\sigma = i$ we define

$$\nu = \mu^2 \delta_{\operatorname{diag}((k+1)^{-1},(k+1)^{-1},i)} + \mu(1-\mu)\delta_{\operatorname{diag}((k+1)^{-1},i,i)} + (1-\mu)\delta_{\operatorname{diag}(i,k^{-1},i)},$$

which, clearly, is a laminate supported in $A^i_{k+1} \cup B^i_{k+1}$, and by (2.3.4), we have that

$$\nu(B^i_{k+1}) \lesssim (k^2 i)^{-1}.$$

If, on the contrary, $\sigma = i-1$, we define $\lambda = \frac{i-1-(k+1)^{-1}}{i-(k+1)^{-1}}$ and using (2.3.5) and that

$$i-1 = \lambda i + (1-\lambda)(k+1)^{-1},$$

we do the following splits:

$$\begin{tikzcd}
& \bullet\, ((k+1)^{-1}, i, i) \in B^i_{k+1} \\
((k+1)^{-1}, i, i-1) \arrow[ur, dash, "\lambda"] \arrow[dr, dash, "1-\lambda"'] & \\
& \bullet\, ((k+1)^{-1}, i, (k+1)^{-1}) \in A^i_{k+1},
\end{tikzcd}$$

$$\begin{tikzcd}
& \bullet\, (i, k^{-1}, i) \in B^i_{k+1} & \\
(i, k^{-1}, i-1) \arrow[ur, dash, "\lambda"] \arrow[dr, dash, "1-\lambda"'] & & \bullet\, (i, (k+1)^{-1}, (k+1)^{-1}) \in A^i_{k+1} \\
& (i, k^{-1}, (k+1)^{-1}) \arrow[ur, dash, "\mu"] \arrow[dr, dash, "1-\mu"'] & \\
& & \bullet\, (i, i, (k+1)^{-1}) \in B^i_{k+1}.
\end{tikzcd}$$

Hence

$$\begin{aligned}
A =& \mu^2 \operatorname{diag}((k+1)^{-1}, (k+1)^{-1}, i-1) + \mu(1-\mu)\lambda \operatorname{diag}((k+1)^{-1}, i, i) \\
&+ \mu(1-\mu)(1-\lambda) \operatorname{diag}((k+1)^{-1}, i, (k+1)^{-1}) \\
&+ (1-\mu)\lambda \operatorname{diag}(i, k^{-1}, i) + (1-\mu)(1-\lambda)\mu \operatorname{diag}(i, (k+1)^{-1}, (k+1)^{-1}) \\
&+ (1-\mu)^2(1-\lambda) \operatorname{diag}(i, i, (k+1)^{-1}),
\end{aligned}$$

and we define

$$\begin{aligned}
\nu =& \mu^2 \delta_{\operatorname{diag}((k+1)^{-1},(k+1)^{-1},i-1)} + \mu(1-\mu)\lambda \delta_{\operatorname{diag}((k+1)^{-1},i,i)} \\
&+ \mu(1-\mu)(1-\lambda)\delta_{\operatorname{diag}((k+1)^{-1},i,(k+1)^{-1})} \\
&+ (1-\mu)\lambda\delta_{\operatorname{diag}(i,k^{-1},i)} + (1-\mu)(1-\lambda)\mu\delta_{\operatorname{diag}(i,(k+1)^{-1},(k+1)^{-1})} \\
&+ (1-\mu)^2(1-\lambda)\delta_{\operatorname{diag}(i,i,(k+1)^{-1})},
\end{aligned}$$

which is a laminate supported in $A^i_{k+1} \cup B^i_{k+1}$. Using (2.3.4) we obtain

$$\nu(B^i_{k+1}) \leqslant 1-\mu^2 \lesssim 1-\mu \leqslant (k^2 i)^{-1},$$

and the proof is complete. □

Now, we can prove the next two lemmas that will give us the laminate ν_A.

Lemma 2.3.4. *Let $i \in \mathbb{N}$, $i \leqslant j$ and $A \in A^i_j$. Then, there exists a laminate ν such that*

- $\overline{\nu} = A$,
- $\operatorname{supp}(\nu) \subset \bigcup_{b=0}^{j-i+1} A^{i+b}_{j+1} \cup B^{j+1}_{j+1}$,
- $\nu(A^{i+b}_{j+1}) \lesssim j^{-2}\frac{i}{(i+b)^3}$, *for* $b \in \{1,\dots,j-i+1\}$,
- $\nu(B^{j+1}_{j+1}) \lesssim j^{-2} i (j+1)^{-2}$.

Lemma 2.3.5. *Let $i \in \mathbb{N}$, $i \leqslant j$ and $A \in B^j_i$. Then, there exists a laminate ν such that*

- $\overline{\nu} = A$,
- $\operatorname{supp}(\nu) \subset A^{j+1}_{j+1} \cup \bigcup_{b=0}^{j-i+1} B^{j+1}_{i+b}$,
- $\nu(A^{j+1}_{j+1}) \lesssim j^{-1}$,
- $\nu(B^{j+1}_i) - \left(\frac{j}{j+1}\right)^2 \lesssim j^{-2} i^{-2}$,
- $\nu(B^{j+1}_{i+b}) \lesssim \big((i+b-1)(j+1)\big)^{-2}$, *for* $b \in \{1,\dots,j-i+1\}$.

As before, we will only prove Lemma 2.3.4 since the proof of Lemma 2.3.5 can be obtained in the same form.

Proof of Lemma 2.3.4. It is enough to construct a family of laminates $\{\nu'_\ell\}_{\ell=1}^{j-i+2}$ such that

i) $\overline{\nu}'_\ell = A$,

ii) $\operatorname{supp}(\nu'_\ell) \subset \bigcup_{b=0}^{\ell-1} A^{i+b}_{j+1} \cup B^{i+\ell-1}_{j+1}$,

iii) $\nu'_\ell(A^{i+b}_{j+1}) \lesssim j^{-2}\frac{i}{(i+b)^3}$, for $b \in \{1,\dots,\ell-1\}$,

iv) $\nu'_\ell(B^{i+\ell-1}_{j+1}) \lesssim j^{-2} i (i+\ell-1)^{-2}\left(1 + j^{-2} 8C \sum_{k=1}^{\ell-1}(i+k-1)^{-2}\right)$,

and define $\nu = \nu'_{j-i+2}$. The constant C is bigger than those that appear in Lemma 2.3.3. Let ν'_1 be the laminate of Lemma 2.3.2; then, ν'_1 satisfies all the conditions. Now, for $1 \leqslant \ell \leqslant j-i+1$, given $\nu'_\ell = \sum_{a=1}^{N_\ell} \lambda_a \delta_{B_a}$ with all the B_a different, define ν_{B_a} as the laminate of Lemma 2.3.3 if $B_a \in B^{i+\ell-1}_{j+1}$ and as δ_{B_a} otherwise. Set

$$\nu'_{\ell+1} = \sum_{a=1}^{N_\ell} \lambda_a \nu_{B_a}.$$

It is immediate that $\nu'_{\ell+1}$ satisfies i), ii), and iii) for $b \in \{1,\dots,\ell-1\}$. Thanks to Corollary 1.0.3, it is a laminate. Hence, we only have to bound $\nu'_{\ell+1}(A^{i+\ell}_{j+1})$ and $\nu'_{\ell+1}(B^{i+\ell}_{j+1})$. We have

$$\nu'_{\ell+1}(A^{i+\ell}_{j+1}) \lesssim \nu'_\ell(B^{i+\ell-1}_{j+1})(i+\ell-1)^{-1} \lesssim j^{-2} i (i+\ell-1)^{-3} \lesssim j^{-2}\frac{i}{(i+\ell)^3}.$$

The first inequality is Lemma 2.3.3 and the second is iv).

For j big enough, $j^{-2}8C\sum_{k=1}^{\infty}(i+k-1)^{-2} \leqslant 1$, so

$$
\begin{aligned}
&1+4C((i+\ell-1)(j+1))^{-2}+\left(1+4C((i+\ell-1)(j+1))^{-2}\right)j^{-2}8C\sum_{k=1}^{\ell-1}(i+k-1)^{-2}\\
&\leqslant 1+j^{-2}8C\sum_{k=1}^{\ell}(i+k-1)^{-2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\nu'_{\ell+1}(B_{j+1}^{i+\ell}) &\leqslant \nu'_{\ell}(B_{j+1}^{i+\ell-1})\left(\left(\frac{i+\ell-1}{i+\ell}\right)^2+C((i+\ell-1)(j+1))^{-2}\right)\\
&\lesssim j^{-2}i(i+\ell)^{-2}\Bigg(1+4C((i+\ell-1)(j+1))^{-2}\\
&\qquad+\left(1+4C((i+\ell-1)(j+1))^{-2}\right)j^{-2}8C\sum_{k=1}^{\ell-1}(i+k-1)^{-2}\Bigg)\\
&\leqslant j^{-2}i(i+\ell)^{-2}\left(1+j^{-2}8C\sum_{k=1}^{\ell}(i+k-1)^{-2}\right).
\end{aligned}
$$

The first inequality is Lemma 2.3.3 and the second is iv). □

From ν_j as in (2.3.3), we construct ν_{j+1} as follows. For each $A \in \operatorname{supp}(\nu_j)$ we define ν_A as the laminate of Lemma 2.3.4 if $A \in \bigcup_{i=1}^{j} A_j^i$, and as the laminate of Lemma 2.3.5 if $A \in \bigcup_{i=1}^{j} B_i^j$. We define ν_{j+1} as

$$\nu_{j+1}=\sum_{k=1}^{N}\lambda_k\nu_{A_k}.$$

It is clear that ν_{j+1} satisfies (a) and (b) of Subsection 2.3.1 (page 70). Pick $\varepsilon>0$ small enough to satisfy $\sum_{a=1}^{\infty}a^{-2+\varepsilon}\leqslant 2$. We will prove by induction on j that ν_{j+1} satisfies (c). Let $C_0>0$ be a constant bigger than the constants appearing in Lemmas 2.3.4 and 2.3.5. Set $j_\varepsilon \in \mathbb{N}$ such that $j_\varepsilon^{\varepsilon}>20C_0$. Define

$$C_j=\max_{i\in\{1,\ldots,j\}}\left\{\nu_j(A_j^i)i^{3-\varepsilon},\nu_j(B_i^j)i^{2-\varepsilon}j^2\right\}\quad \text{for } j\leqslant j_\varepsilon,$$

and

$$C_{j+1}=C_j(1+12C_0j^{-2})\quad \text{for } j\geqslant j_\varepsilon.$$

It is clear that $\sup_{j\in\mathbb{N}}C_j<\infty$, and we have (c) for $j\leqslant j_\varepsilon$.

Then for $j\geqslant j_\varepsilon$ and for each $i\in\{1,\ldots,j\}$ we know that the matrices in A_{j+1}^i can only come from $\bigcup_{a=1}^{i}A_j^a$, and the matrices in B_i^{j+1} can only come from $\bigcup_{a=1}^{i}B_a^j$; therefore using respectively the bounds of Lemmas 2.3.4 and 2.3.5 we obtain

$$
\begin{aligned}
\nu_{j+1}(A_{j+1}^i) &\leqslant \nu_j(A_j^i)+\sum_{a=1}^{i-1}\nu_j(A_j^a)C_0j^{-2}\frac{a}{i^3}\leqslant C_j\left(i^{-3+\varepsilon}+\sum_{a=1}^{i-1}C_0j^{-2}a^{-2+\varepsilon}i^{-3}\right)\\
&\leqslant C_j(1+2C_0j^{-2})i^{-3+\varepsilon}\leqslant C_{j+1}i^{-3+\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
\nu_{j+1}(B_i^{j+1}) &\leqslant \nu_j(B_i^j)\left(\left(\frac{j}{j+1}\right)^2 + C_0 j^{-2} i^{-2}\right) + \sum_{a=1}^{i-1} \nu_j(B_a^j) C_0((i-1)(j+1))^{-2} \\
&\leqslant C_j \left(i^{-2+\varepsilon}(j+1)^{-2} + C_0 i^{-4+\varepsilon} j^{-4} + 4C_0 i^{-2} j^{-2}(j+1)^{-2} \sum_{a=1}^{i-1} a^{-2+\varepsilon}\right) \\
&\leqslant C_j i^{-2+\varepsilon}(j+1)^{-2}(1+12C_0 j^{-2}) = C_{j+1} i^{-2+\varepsilon}(j+1)^{-2}.
\end{aligned}
$$

For $i = j+1$, since the matrices in $A_{j+1}^{j+1} \cup B_{j+1}^{j+1}$ can come from any matrix in the support of ν_j we get

$$
\begin{aligned}
\nu_{j+1}(A_{j+1}^{j+1}) &\leqslant \sum_{a=1}^{j} \left[\nu_j(A_j^a) C_0 j^{-2} \frac{a}{(j+1)^3} + \nu_j(B_a^j) C_0 j^{-1} \right] \\
&\leqslant C_j C_0 \sum_{a=1}^{i-1} \left[j^{-2} a^{-2+\varepsilon}(j+1)^{-3} + a^{-2+\varepsilon} j^{-3}\right] \\
&\leqslant C_j 20 C_0 (j+1)^{-3} \leqslant C_{j+1}(j+1)^{-3+\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
\nu_{j+1}(B_{j+1}^{j+1}) &\leqslant \sum_{a=1}^{j} \left[\nu_j(A_j^a) C_0 j^{-2} a (j+1)^{-2} + \nu_j(B_a^j) C_0 j^{-2}(j+1)^{-2} \right] \\
&\leqslant C_j \left(C_0 \sum_{a=1}^{i-1} \left[j^{-2} a^{-2+\varepsilon}(j+1)^{-2} + a^{-2+\varepsilon} j^{-4}(j+1)^{-2}\right]\right) \\
&\leqslant C_j 20 C_0 (j+1)^{-4} \leqslant C_{j+1}(j+1)^{-4+\varepsilon},
\end{aligned}
$$

and the proof of (a)–(c) is completed.

2.3.2 Proof of Theorem 2.3.1

The sets that we define next are the key of the proof, which consists of constructing laminates ν_j supported in

$$
\bigcup_{i=1}^{j} \left(A_j^i \cup B_i^j \cup \bigcup_{a=m_2+1}^{n-m_1-1} \left(S_{i,j}^a \cup S_{j,i}^a \right) \right),
$$

and then approximate those laminates by homeomorphisms using Proposition 1.3.5.

Given $A \in \mathbb{R}^{n\times n}$, let $\sigma_1 \leqslant \cdots \leqslant \sigma_n$ be its singular values. For $i, k \in \mathbb{N}\setminus\{0\}$ we define the following sets in the case $m_1 + m_2 \leqslant n-1$:

$$
A_k^i = \left\{ \begin{array}{c} A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \dfrac{(k+1)^{-2}}{4} \text{ for } j \in \{1,\dots,n-m_1\} \text{ and} \\ i - \dfrac{1}{4} < \sigma_j < i + \dfrac{5}{4} \text{ for } j \in \{n-m_1+1,\dots,n\} \end{array} \right\},
$$

$$
B_k^i = \left\{ \begin{array}{c} A \in \Gamma_+ : (k+1)^{-1} - \dfrac{(k+1)^{-2}}{4} < \sigma_j < k^{-1} + \dfrac{k^{-2}}{4} \text{ for } j \in \{1,\dots,m_2\} \\ \text{and } |\sigma_j - i - 1| < \dfrac{1}{4} \text{ for } j \in \{m_2+1,\dots,n\} \end{array} \right\},
$$

and for $a \in \{m_2+1, \dots, n-m_1-1\}$ we define

$$S^a_{k,i} = \left\{ \begin{array}{c} A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \dfrac{(k+1)^{-2}}{4} \text{ for } j \in \{1, \dots, a\} \text{ and} \\ |\sigma_j - i - 1| < \dfrac{1}{4} \text{ for } j \in \{a+1, \dots, n\} \end{array} \right\}.$$

We will only prove the theorem in the previous case, since, in the case $m_1 + m_2 \geqslant n$, the proof is the same using the next sets instead of the above:

$$A^i_k = \left\{ \begin{array}{l} A \in \Gamma_+ : |\sigma_j - (k+1)^{-1}| < \dfrac{(k+1)^{-2}}{4} \text{ for } j \in \{1, \dots, n-m_1\},\ \dfrac{1}{2} < \sigma_j < 2 \\ \text{for } j \in \{n-m_1+1, \dots, m_2+1\} \text{ and } i - \dfrac{1}{4} < \sigma_j < i + \dfrac{5}{4} \text{ for } j \in \{m_2+2, \dots, n\} \end{array} \right\},$$

and

$$B^i_k = \left\{ \begin{array}{l} A \in \Gamma_+ : (k+1)^{-1} - \dfrac{(k+1)^{-2}}{4} < \sigma_j < k^{-1} + \dfrac{k^{-2}}{4} \text{ for } j \in \{1, \dots, n-m_1-1\}, \\ \dfrac{1}{2} < \sigma_j < 2 \text{ for } j \in \{n-m_1, \dots, m_2\} \text{ and } |\sigma_j - i - 1| < \dfrac{1}{4} \text{ for } j \in \{m_2+1, \dots, n\} \end{array} \right\}.$$

The most important case is when $m_1 + m_2 = n-1$, where we have $S^a_{k,i} = \emptyset$, and, therefore, the proof is much simpler. When $m_1 + m_2 < n-1$ the sets $S^a_{k,i}$ constitute an interpolation between A^i_k and B^i_k. We recommend the reader to focus on the case $m_1 + m_2 = n-1$ in a first read.

In order to write all the lemmas in a form that include all the cases, we recall the definition

$$m'_i = \begin{cases} m_i & \text{if } m_1 + m_2 \leqslant n-1, \\ m_i - (m_1 + m_2 - n + 1) & \text{if } m_1 + m_2 \geqslant n, \end{cases}$$

for $i \in \{1, 2\}$, and we define

$$n' = \begin{cases} n & \text{if } m_1 + m_2 \leqslant n-1, \\ 2n - m_1 - m_2 - 1 & \text{if } m_1 + m_2 \geqslant n. \end{cases}$$

In the next lemma we construct a laminate supported in $A^i_{k+1} \cup B^i_{k+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{k+1,i}$ whose barycenter is a given matrix in A^i_k. We provide the proof in the case $m_1+m_2 \leqslant n-1$. If $m_1+m_2 \geqslant n$, we fix the eigenvalues $\{\sigma_j\}_{j=n-m_1+1}^{m_2+1}$, which are the eigenvalues in $(\frac{1}{2}, 2)$, and we construct the same laminate as in the first case over the other eigenvalues, i.e., given $A = \operatorname{diag}(\sigma_1, \dots, \sigma_n) \in A^i_k$, let $A' = \operatorname{diag}(\sigma_1, \dots, \sigma_{n-m_1}, \sigma_{m_2+2}, \dots, \sigma_n) \in \mathbb{R}^{n' \times n'}$ and apply Lemma 2.3.6 with n', m'_1 and m'_2 to get the laminate $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M'_\ell}$, where

$$M'_\ell = \operatorname{diag}(s^\ell_1, \dots, s^\ell_{n'}).$$

For $\ell = 1, \dots, N$, define

$$M_\ell = \operatorname{diag}(s^\ell_1, \dots, s^\ell_{n-m_1}, \sigma_{n-m_1+1}(A), \dots, \sigma_{m_2+1}(A), s^\ell_{n-m_1+1}, \dots, s^\ell_{n'}).$$

So, in the case $m_1 + m_2 \geqslant n$ we would work with the laminate $\sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$.

The bounds of $S^a_{k,i}$ along the paper only make sense when $m_1 + m_2 < n$; otherwise, the $S^a_{k,i}$ are empty.

Lemma 2.3.6. *Let $A \in A^i_k$. Then there exists a laminate of finite order $\nu = \sum_{\ell=1}^N \lambda_\ell \delta_{M_\ell}$ such that*

a) $\overline{\nu} = A$,

b) $\operatorname{supp}(\nu) \subset A^i_{k+1} \cup B^i_{k+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{k+1,i}$,

c) $\nu(A^i_{k+1}) \leqslant 1$,

d) $\nu(B^i_{k+1}) \lesssim \left(k^2 i\right)^{m_1'+m_2'-n'}$,

e) $\nu(S^a_{k+1,i}) \lesssim \left(k^2 i\right)^{m_1'+a-n'}$ *for* $a \in \{m_2+1,\dots,n-m_1-1\}$.

f) $M_1 \in A^i_{k+1}$, $|A - M_1| \leqslant k^{-2}$, $|A^{-1} - M_1^{-1}| \lesssim 1$ *and* $1 - \lambda_1 \lesssim k^{-2} i^{-1}$.

Proof. Since we give the proof in the case $m_1 + m_2 \leqslant n-1$, we have $m_1' = m_1$ and $m_2' = m_2$. Without loss of generality we can assume that A is a diagonal matrix, hence $A = \operatorname{diag}(\sigma_1,\dots,\sigma_n)$ with $|\sigma_j - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}$ for $j \in \{1,\dots,n-m_1\}$ and $i - \frac{1}{4} < \sigma_j < i + \frac{5}{4}$ for $j \in \{n-m_1+1,\dots,n\}$.

Let $b = \operatorname{Card}\{j \in \{n-m_1+1,\dots,n\} : \sigma_j > i + \frac{3}{4}\}$.

We shall construct a family $\{B_{\ell,j}\}_{\substack{\ell=0,\dots,n-b\\ j=0,\dots,2^\ell-1}}$ in Γ_+ and a family $\{\lambda_{\ell,j}\}_{\substack{\ell=0,\dots,n-b\\ j=0,\dots,2^\ell-1}}$ in $[0,1]$ by finite induction on ℓ.

Let $B_{0,0} = A$, $\lambda_{0,0} = 1$ and for $0 \leqslant \ell \leqslant n-b-1$, $0 \leqslant j \leqslant 2^\ell - 1$, we assume that $\{B_{\ell,j}\}_{j=0}^{2^\ell-1}$ and $\{\lambda_{\ell,j}\}_{j=0}^{2^\ell-1}$ have been defined, $B_{\ell,j}$ are diagonal, $\lambda_{\ell,j} \geqslant 0$,

$$\sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} = 1, \qquad B_{0,0} = \sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} B_{\ell,j} \tag{2.3.6}$$

$$\sum_{j=0}^{2^\ell-1} \lambda_{\ell,j} \delta_{B_{\ell,j}} \in \mathscr{L}(\mathbb{R}^{n\times n}), \tag{2.3.7}$$

and when we let

$$\beta^1_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{1,\dots,n-m_1\} : |\left(B_{\ell,j}\right)_{\alpha,\alpha} - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}\right\},$$

$$\beta^2_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{1,\dots,n-m_1\} : |\left(B_{\ell,j}\right)_{\alpha,\alpha} - (k+2)^{-1}| < \frac{(k+2)^{-2}}{4}\right\},$$

$$\beta^3_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{1,\dots,n-m_1\} : |\left(B_{\ell,j}\right)_{\alpha,\alpha} - i - 1| < \frac{1}{4}\right\},$$

$$\gamma^1_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{n-m_1+1,\dots,n-b\} : i - \frac{1}{4} < \left(B_{\ell,j}\right)_{\alpha,\alpha} \leqslant i + \frac{3}{4}\right\},$$

$$\gamma^2_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{n-m_1+1,\dots,n-b\} : |\left(B_{\ell,j}\right)_{\alpha,\alpha} - (k+2)^{-1}| < \frac{(k+2)^{-2}}{4}\right\},$$

$$\gamma^3_{\ell,j} := \operatorname{Card}\left\{\alpha \in \{n-m_1+1,\dots,n-b\} : |\left(B_{\ell,j}\right)_{\alpha,\alpha} - i - 1| < \frac{1}{4}\right\},$$

then

$$\beta^1_{\ell,j}+\beta^2_{\ell,j}+\gamma^2_{\ell,j}\leqslant n-m_1, \tag{2.3.8}$$

$$\beta^3_{\ell,j}+\gamma^1_{\ell,j}+\gamma^3_{\ell,j}\leqslant n-m_2-b, \tag{2.3.9}$$

$$\beta^1_{\ell,j}+\beta^2_{\ell,j}+\beta^3_{\ell,j}+\gamma^1_{\ell,j}+\gamma^2_{\ell,j}+\gamma^3_{\ell,j}=n-b, \tag{2.3.10}$$

$$\lambda_{\ell,j}\leqslant\left(\frac{2}{i(k+1)^2}\right)^{\beta^3_{\ell,j}}\left(\frac{2}{i}\right)^{\gamma^2_{\ell,j}}, \tag{2.3.11}$$

$$B_{\ell,0}=\operatorname{diag}\left(\underbrace{(k+2)^{-1},\dots,(k+2)^{-1}}_{\min\{\ell,n-m_1\}},\sigma_{\min\{\ell,n-m_1\}+1},\dots,\sigma_n\right) \tag{2.3.12}$$

and

$$\lambda_{\ell,0}=\prod_{j=1}^{\min\{\ell,n-m_1\}}\frac{i+1-\sigma_j}{i+1-(k+2)^{-1}}. \tag{2.3.13}$$

Moreover, for those $B_{\ell,j}\notin A^i_{k+1}\cup B^i_{k+1}$, we have

$$\beta^1_{\ell,j}+\gamma^1_{\ell,j}\leqslant n-\ell-b. \tag{2.3.14}$$

Since (2.3.10) holds and $m_1+m_2\leqslant n-1$, we see that (2.3.8) and (2.3.9) cannot be equalities at the same time.

We observe that the sets appearing in the definitions of $\beta^a_{\ell,j},\gamma^a_{\ell,j}$ for $a=1,2,3$ are pairwise disjoint; we also see that

$$\beta^2_{0,0}=\gamma^2_{0,0}=\beta^3_{0,0}=\gamma^3_{0,0}=0,\quad \beta^1_{0,0}=n-m_1,\qquad \gamma^1_{0,0}=m_1-b.$$

Now, we start with the induction. If $\beta^2_{\ell,j}+\gamma^2_{\ell,j}=n-m_1$, then $B_{\ell,j}\in A^i_{k+1}$, if $\beta^3_{\ell,j}+\gamma^3_{\ell,j}=n-m_2-b$, then $B_{\ell,j}\in B^i_{k+1}$, and, if $a:=\beta^2_{\ell,j}+\gamma^2_{\ell,j}<n-m_1$, $\beta^3_{\ell,j}+\gamma^3_{\ell,j}<n-m_2-b$ and $\beta^1_{\ell,j}+\gamma^1_{\ell,j}=0$ then $B_{\ell,j}\in S^a_{k+1,i}$.

Now, for $j=0,\dots,2^{\ell+1}-1$ we construct $B_{\ell+1,j}$ and $\lambda_{\ell+1,j}$.

If $B_{\ell,\lfloor\frac{j}{2}\rfloor}\in A^i_{k+1}\cup B^i_{k+1}\cup\bigcup_{a=n-m_1+1}^{m_2-1}S^a_{k+1,i}$ we define $B_{\ell+1,j}=B_{\ell,\lfloor\frac{j}{2}\rfloor}$ and

$$\lambda_{\ell+1,j}=\begin{cases}\lambda_{\ell,\frac{j}{2}}, & \text{if } j \text{ is even,}\\ 0, & \text{if } j \text{ is odd.}\end{cases}$$

So it is clear that (2.3.8)–(2.3.11) are satisfied.

In the case $B_{\ell,\lfloor\frac{j}{2}\rfloor} \notin A^i_{k+1} \cup B^i_{k+1} \cup \bigcup_{a=n-m_1+1}^{m_2-1} S^a_{k+1,i}$, we have

$$\beta^2_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor} < n - m_1, \tag{2.3.15}$$

$$\beta^3_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor} < n - m_2 - b, \tag{2.3.16}$$

and

$$\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} > 0, \tag{2.3.17}$$

and, we divide the construction of $B_{\ell+1,j}$ into two cases, according to whether (2.3.18) or (2.3.19) holds. We observe that if $B_{\ell,0} \in A^i_{k+1} \cup B^i_{k+1} \cup \bigcup_{a=n-m_1+1}^{m_2-1} S^a_{k+1,i}$ then $B_{\ell,0} \in A^i_{k+1}$, which happens if and only if $\ell < n - m_1$. Hence, if $\ell \geqslant n - m_1$, (2.3.12) and (2.3.13) are satisfied, whereas if $\ell < n - m_1$ then (2.3.18) is satisfied for $j = 0$.

If

$$\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} > 0 \text{ and } \beta^3_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor} < n - m_2 - b, \tag{2.3.18}$$

let $\alpha \in \{1, \dots, n - m_1\}$ be the smallest number such that $|(B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha} - (k+1)^{-1}| < \frac{(k+1)^{-2}}{4}$. Then, we define

$$B_{\ell+1,j} = \begin{cases} B_{\ell,\lfloor\frac{j}{2}\rfloor} + \operatorname{diag}\Big(\underbrace{0,\dots,0}_{\alpha-1}, (k+2)^{-1} - (B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}, \underbrace{0,\dots,0}_{n-\alpha}\Big), & \text{if } j \text{ is even,} \\ B_{\ell,\lfloor\frac{j}{2}\rfloor} + \operatorname{diag}\Big(\underbrace{0,\dots,0}_{\alpha-1}, i+1 - (B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}, \underbrace{0,\dots,0}_{n-\alpha}\Big), & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\lambda_{\ell+1,j} = \begin{cases} \frac{i+1-(B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}}{i+1-(k+2)^{-1}} \lambda_{\ell,\lfloor\frac{j}{2}\rfloor} \leqslant \lambda_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is even,} \\ \frac{(B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}-(k+2)^{-1}}{i+1-(k+2)^{-1}} \lambda_{\ell,\lfloor\frac{j}{2}\rfloor} \leqslant \frac{2}{i(k+1)^2} \lambda_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is odd.} \end{cases}$$

Then

$$B_{\ell,\frac{j}{2}} = \frac{i+1-(B_{\ell,\frac{j}{2}})_{\alpha,\alpha}}{i+1-(k+2)^{-1}} B_{\ell+1,j} + \frac{(B_{\ell,\frac{j}{2}})_{\alpha,\alpha} - (k+2)^{-1}}{i+1-(k+2)^{-1}} B_{\ell+1,j+1} \text{ for } j \in \{0, \dots, 2^{\ell+1}-1\} \text{ even,}$$

and hence

$$\sum_{j=0}^{2^{\ell+1}-1} \lambda_{\ell+1,j} \delta_{B_{\ell+1,j}} \in \mathscr{L}(\mathbb{R}^{n\times n}),$$

$$\beta^1_{\ell+1,j} = \beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} - 1,$$

$$\beta^2_{\ell+1,j} = \begin{cases} \beta^2_{\ell,\lfloor\frac{j}{2}\rfloor} + 1, & \text{if } j \text{ is even,} \\ \beta^2_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is odd,} \end{cases}$$

$$\beta^3_{\ell+1,j} = \begin{cases} \beta^3_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is even,} \\ \beta^3_{\ell,\lfloor\frac{j}{2}\rfloor}+1, & \text{if } j \text{ is odd,} \end{cases}$$

$$\gamma^1_{\ell+1,j} = \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor}, \quad \gamma^2_{\ell+1,j} = \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor} \quad \text{and } \gamma^3_{\ell+1,j} = \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor}.$$

Therefore, (2.3.6)–(2.3.14) are satisfied for $\ell+1$.

If

$$\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} = 0 \text{ or } \beta^3_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor} = n - m_2 - b \tag{2.3.19}$$

instead of (2.3.18), we claim that then we have $\gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} > 0$ and

$$\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \beta^2_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor} < n - m_1.$$

Indeed, if $\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} = 0$, it is clear thanks to (2.3.15) and (2.3.17), and if

$$\beta^3_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor} = n - m_2 - b,$$

we have

$$\beta^1_{\ell,\lfloor\frac{j}{2}\rfloor} + \beta^2_{\ell,\lfloor\frac{j}{2}\rfloor} + \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor} = m_2 < n - m_1,$$

and by (2.3.16) we obtain $\gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} > 0$. Therefore there exists $\alpha \in \{n-m_1+1,\dots,n-b\}$ such that

$$i - \frac{1}{4} < (B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha} \leqslant i + \frac{3}{4}.$$

Then, we define

$$B_{\ell+1,j} = \begin{cases} B_{\ell,\lfloor\frac{j}{2}\rfloor} + \operatorname{diag}\Big(\underbrace{0,\dots,0}_{\alpha-1}, (k+2)^{-1} - (B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}, \underbrace{0,\dots,0}_{n-\alpha}\Big), & \text{if } j \text{ is even,} \\ B_{\ell,\lfloor\frac{j}{2}\rfloor} + \operatorname{diag}\Big(\underbrace{0,\dots,0}_{\alpha-1}, i+1 - (B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}, \underbrace{0,\dots,0}_{n-\alpha}\Big), & \text{if } j \text{ is odd,} \end{cases}$$

$$\lambda_{\ell+1,j} = \begin{cases} \frac{i+1-(B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}}{i+1-(k+2)^{-1}} \lambda_{\ell,\lfloor\frac{j}{2}\rfloor} \leqslant \frac{2}{i}\lambda_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is even,} \\ \frac{(B_{\ell,\lfloor\frac{j}{2}\rfloor})_{\alpha,\alpha}-(k+2)^{-1}}{i+1-(k+2)^{-1}} \lambda_{\ell,\lfloor\frac{j}{2}\rfloor} \leqslant \lambda_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is odd.} \end{cases}$$

Then

$$\sum_{j=0}^{2^{\ell+1}-1} \lambda_{\ell+1,j}\delta_{B_{\ell+1,j}} \in \mathscr{L}(\mathbb{R}^{n\times n}),$$

$$\beta^1_{\ell+1,j} = \beta^1_{\ell,\lfloor\frac{j}{2}\rfloor}, \quad \beta^2_{\ell+1,j} = \beta^2_{\ell,\lfloor\frac{j}{2}\rfloor}, \quad \beta^3_{\ell+1,j} = \beta^3_{\ell,\lfloor\frac{j}{2}\rfloor},$$

$$\gamma^1_{\ell+1,j} = \gamma^1_{\ell,\lfloor\frac{j}{2}\rfloor} - 1,$$

$$\gamma^2_{\ell+1,j} = \begin{cases} \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor} + 1, & \text{if } j \text{ is even,} \\ \gamma^2_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is odd,} \end{cases}$$

and

$$\gamma^3_{\ell+1,j} = \begin{cases} \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor}, & \text{if } j \text{ is even,} \\ \gamma^3_{\ell,\lfloor\frac{j}{2}\rfloor} + 1, & \text{if } j \text{ is odd.} \end{cases}$$

Therefore, (2.3.6), (2.3.7), (2.3.8), (2.3.9), (2.3.10), (2.3.11) and (2.3.14) are satisfied for $\ell+1$.

Here ends the inductive construction of $\{B_{\ell,j}\}_{\substack{\ell=0,\dots,n-b \\ j=0,\dots,2^\ell-1}}$. With this, we define

$$N = 2^{n-b}, \quad \lambda_\ell = \lambda_{n-b,\ell-1}, \quad M_\ell = B_{n-b,\ell-1} \text{ for } \ell = 1,\dots,2^{n-b} \quad \text{and } \nu := \sum_{j=1}^{2^{n-b}} \lambda_j \delta_{M_j},$$

which is a laminate by (2.3.7), and by (2.3.10) and (2.3.14) it is supported in $A^i_{k+1} \cup B^i_{k+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{k+1,i}$. We shall check properties *a)–f)*. Property *a)* comes from (2.3.6), *b)* is by (2.3.14), and *c)* is obvious.

Now, we use (2.3.11) to bound the mass of ν in the different sets.

Since $\gamma^3_{n-b,j} \leqslant m_1 - b$ and the matrices $B_{n-b,j}$ in B^i_{k+1} are those such that $\beta^3_{n-b,j} + \gamma^3_{n-b,j} = n - m_2 - b$, we have $\beta^3_{n-b,j} \geqslant n - m_1 - m_2$ and, thanks to (2.3.11), we also get

$$\nu(B^i_{k+1}) = \sum_{j:\beta^3_{n-b,j}+\gamma^3_{n-b,j}=n-m_2-b} \lambda_{n-b,j} \lesssim \left(\frac{2}{i(k+1)^2}\right)^{n-m_1-m_2} \lesssim \left(k^2 i\right)^{m_1+m_2-n}.$$

So *d)* is proved. Now we use that $\sum_{l=1}^3 \beta^l_{n-b,j} = n - m_1$ and that for $a \in \{m_2+1,\dots,n-m_1-1\}$ the matrices $B_{n-b,j}$ in $S^a_{k+1,i}$ are those such that $\beta^2_{n-b,j} + \gamma^2_{n-b,j} = a$, $\beta^1_{n-b,j} = \gamma^1_{n-b,j} = 0$, so

$$\beta^3_{n-b,j} = n - m_1 - \beta^2_{n-b,j} = n - m_1 + \gamma^2_{n-b,j} - a \geqslant n - m_1 - a,$$

hence, using (2.3.11) we obtain

$$\nu(S^a_{k+1,i}) = \sum_{j:\beta^2_{n-b,j}+\gamma^2_{n-b,j}=a} \lambda_{n-b,j} \lesssim \left(\frac{2}{i(k+1)^2}\right)^{n-m_1-a} \lesssim \left(k^2 i\right)^{m_1+a-n}.$$

Therefore we have *e)*. Finally, we prove *f)*. Thanks to (2.3.12) we get

$$M_1 = \operatorname{diag}\Bigg(\underbrace{(k+2)^{-1},\dots,(k+2)^{-1}}_{n-m_1}, \sigma_{n-m_1+1},\dots,\sigma_n\Bigg)$$

and due to (2.3.13) we obtain

$$\lambda_1 = \prod_{j=1}^{n-m_1} \frac{i+1-\sigma_j}{i+1-(k+2)^{-1}}.$$

Therefore, using $k^{-1}-(k+1)^{-1}\leqslant k^{-2}$ we get

$$|A-M_1|\leqslant k^{-2},\quad |A^{-1}-M^{-1}|\lesssim 1$$

and, noting that $n-m_1\geqslant 2$ we also obtain

$$\begin{aligned}1-\lambda_1 &\leqslant 1-\left(\frac{i+1-(k+1)^{-1}-\frac{(k+1)^{-2}}{4}}{i+1-(k+2)^{-1}}\right)^{n-m_1}\\ &\lesssim \frac{\left((k+1)^{-1}+\frac{(k+1)^{-2}}{4}\right)^{n-m_1-1}-(k+2)^{m_1-n+1}}{i}\lesssim k^{-2}i^{-1}.\end{aligned}$$

The second estimate is because of the numerical inequality

$$1-\left(\frac{a-x}{a-y}\right)^k\lesssim\frac{x^{k-1}-y^{k-1}}{a}\text{ when }0<y<x<a,\ k\in\mathbb{N}.$$

Hence, the proof is finished. □

In the next two lemmas we give the laminates starting in B_k^i and $S_{k,i}^a$. We will not show them, since their construction mimic that of Lemma 2.3.6. The main difference in the proofs of Lemmas 2.3.7 and 2.3.8 with the one of Lemma 2.3.6 is that we split the eigenvalues σ close to $i+1$ in $(k+1)^{-1}$ and $\sigma+1$. Also, in the proof of Lemma 2.3.7 we start splitting the eigenvalues close to $i+1$ and we do not split the eigenvalues in $\left((k+1)^{-1}-\frac{(k+1)^{-2}}{4},(k+1)^{-1}+\frac{(k+1)^{-2}}{4}\right)$.

Lemma 2.3.7. *Let $A\in B_k^i$. Then there exists a laminate of finite order $\nu=\sum_{\ell=1}^N\lambda_\ell\delta_{M_\ell}$ such that*

- $\overline{\nu}=A$,
- $\operatorname{supp}(\nu)\subset A_k^{i+1}\cup B_k^{i+1}\cup\bigcup_{a=m_2+1}^{n-m_1-1}S_{k,i+1}^a$,
- $\nu(A_k^{i+1})\lesssim i^{m_1'+m_2'-n'}$,
- $\nu(B_k^{i+1})-\left(\frac{i+2}{i+3}\right)^{n'-m_2'}\lesssim(ki)^{-2}$,
- $\nu(S_{k,i+1}^a)\lesssim i^{m_2'-a}$ *for* $a\in\{m_2+1,\dots,n-m_1-1\}$,
- $M_1\in B_k^{i+1}$, $|A-M_1|=1$, $|A^{-1}-M_1^{-1}|\leqslant i^{-2}$ *and* $1-\lambda_1\lesssim i^{-1}$.

Lemma 2.3.8. *Let $a_0\in\{m_2+1,\dots,n-m_1-1\}$ and $A\in S_{k,i}^{a_0}$. Then there exists a laminate of finite order $\nu=\sum_{\ell=1}^N\lambda_\ell\delta_{M_\ell}$ such that*

- $\overline{\nu}=A$,
- $\operatorname{supp}(\nu)\subset A_{k+1}^{i+1}\cup B_{k+1}^{i+1}\cup\bigcup_{a=m_2+1}^{n-m_1-1}S_{k+1,i+1}^a$,
- $\nu(A_{k+1}^{i+1})\lesssim i^{a_0+m_1-n}$,
- $\nu(B_{k+1}^{i+1})\lesssim(k^2i)^{m_2-a_0}$,

- $\nu(S^a_{k+1,i+1}) \lesssim (k^2 i)^{a-a_0}$ *if* $a \in \{m_2+1,\dots,a_0-1\}$,
- $\nu(S^{a_0}_{k+1,i+1}) - \left(\frac{i+2}{i+3}\right)^{n-a_0} \lesssim (ki)^{-2}$,
- $\nu(S^a_{k+1,i+1}) \lesssim i^{a_0-a}$ *if* $a \in \{a_0+1,\dots,n-m_1-1\}$,
- $\sum_{\ell=1}^N \lambda_\ell |A-M_\ell| \lesssim 1$,
- $\frac{1}{\det(A)} \sum_{\ell=1}^N \lambda_\ell \det(M_\ell)|A^{-1}-M_\ell^{-1}| \lesssim 1$.

In the proof of the last lemma, besides adapting the proof of Lemma 2.3.6, we follow the proof of *h)* and *i)* of Lemma 2.3.9 below to show the last two items.

In the next lemma we put together Lemmas 2.3.6, 2.3.7 and 2.3.8 to construct laminates whose support is in the set in which we are interested. Again, all the bounds of $S^a_{j+1,i}$ have sense if and only if $m_1+m_2 \leqslant n-1$; otherwise these sets are empty.

Lemma 2.3.9. *Let $i, j \in \mathbb{N}$, $i \leqslant j$, and $A \in A^i_j$. Then there exists a laminate $\nu = \sum_{k=1}^N \lambda_k \delta_{M_k}$ such that*

a) $\overline{\nu} = A$,

b) $\operatorname{supp}(\nu) \subset \bigcup_{b=0}^{j-i+1} A^{i+b}_{j+1} \cup B^{j+1}_{j+1} \cup \bigcup_{b=0}^{j-i+1} \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i+b}$,

c) $\nu(A^i_{j+1}) \leqslant 1$,

d) $\nu(A^{i+b}_{j+1}) \lesssim j^{2(m_1'+m_2'-n')} \frac{i^{m_1'}}{(i+b)^{m_1'+2}}$, *for* $b \in \{1,\dots,j-i+1\}$,

e) $\nu(B^{j+1}_{j+1}) \lesssim j^{2m_1'+3m_2'-3n'} i^{m_1'}$,

f) $\nu(S^a_{j+1,i}) \lesssim (j^2 i)^{m_1'+a-n'}$,

g) $\nu(S^a_{j+1,i+b}) \lesssim j^{2(m_1'+m_2'-n')} i^{m_1'} (i+b)^{2m_2'-a-n'}$ *for* $a \in \{m_2+1,\dots,n-m_1-1\}$ *and* $b \in \{1,\dots,j-i+1\}$,

h) $\sum_{k=1}^N \lambda_k |A-M_k| \lesssim j^{-2}$,

i) $\frac{1}{\det(A)} \sum_{k=1}^N \lambda_k \det(M_k)|A^{-1}-M_k^{-1}| \lesssim 1$.

Proof. Let C be a constant bigger than those in Lemma 2.3.7. It is enough construct a sequence of laminates $\{\nu_\ell\}_{\ell=1}^{j-i+2}$ such that

1) $\overline{\nu}_\ell = A$,

2) $\operatorname{supp}(\nu_\ell) \subset \bigcup_{b=0}^{\ell-1} A^{i+b}_{j+1} \cup B^{i+\ell-1}_{j+1} \cup \bigcup_{b=0}^{\ell-1} \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i+b}$,

3) $\nu_\ell(A^{i+b}_{j+1}) \leqslant j^{2(m_1'+m_2'-n')} \frac{i^{m_1'}}{(i+b)^{m_1'+2}}$, for $b \in \{1,\dots,\ell-1\}$,

4) $\nu_\ell(B^{i+\ell-1}_{j+1}) \lesssim j^{2(m_1'+m_2'-n')} i^{m_1'} (i+\ell+1)^{m_2'-n'} \left(1 + j^{-2} \sum_{k=1}^{\ell-1} 2^n C (i+k)^{-2}\right)$,

5) $\nu_\ell(S^a_{j+1,i}) \lesssim (j^2 i)^{m_1'+a-n'}$,

6) $\nu_\ell(S^a_{j+1,i+b}) \lesssim j^{2(m_1'+m_2'-n')} i^{m_1'} (i+b)^{2m_2'-a-n'}$ for $a \in \{m_2+1,\dots,n-m_1-1\}$ and $b \in \{1,\dots,\ell-1\}$,

7) $M_1 \in A^i_{j+1} \cap \operatorname{supp}(\nu_\ell)$ such that $|A-M_1| \leq j^{-2}$, $|A^{-1}-M_1^{-1}| \lesssim 1$, $1-\nu_\ell(M_1) \lesssim j^{-2} i^{-1}$, with M_1 being the one of Lemma 2.3.6,

and prove later *h)* and *i)*.

Let ν_1 be the laminate of Lemma 2.3.6, which satisfies

- $\overline{\nu}_1 = A$,
- $\operatorname{supp}(\nu_1) \subset A^i_{j+1} \cup B^i_{j+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i}$,
- $\nu_1(B^i_{j+1}) \lesssim \left(j^2 i\right)^{m_1'+m_2'-n'}$,
- $\nu_1(S^a_{j+1,i}) \lesssim \left(j^2 i\right)^{m_1'+a-n'}$ for $a \in \{m_2+1,\dots,n-m_1-1\}$,
- $\exists M \in A^i_{j+1} \cap \operatorname{supp}(\nu_1)$ such that $|A-M| \leq j^{-2}$, $|A^{-1}-M^{-1}| \lesssim 1$ and $1-\nu_1(M) \lesssim j^{-2} i^{-1}$.

Therefore, denoting by M_1 the matrix M, we get that ν_1 satisfies 1)–7).

Now, we proceed by induction and assume that ν_ℓ has been constructed with the properties 1)–7). For each $B \in \operatorname{supp}(\nu_\ell) \cap B^{i+\ell-1}_{j+1}$, let ν_B be the laminate given by Lemma 2.3.7, which satisfies

- $\overline{\nu}_B = B$,
- $\operatorname{supp}(\nu_B) \subset A^{i+\ell}_{j+1} \cup B^{i+\ell}_{j+1} \cup \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i+\ell}$,
- $\nu_B(A^{i+\ell}_{j+1}) \lesssim (i+\ell)^{m_1'+m_2'-n'}$,
- $\nu_B(B^{i+\ell}_{j+1}) - \left(\frac{i+\ell+1}{i+\ell+2}\right)^{n'-m_2'} \leq C(i+\ell)^{-2} j^{-2}$,
- $\nu_B(S^a_{j+1,i+\ell}) \lesssim (i+\ell)^{m_2'-a}$ for $a \in \{m_2+1,\dots,n-m_1-1\}$.

We define

$$\nu_{\ell+1} = \nu_\ell + \sum_{B \in \operatorname{supp}(\nu_\ell) \cap B^{i+\ell-1}_{j+1}} \nu_\ell(B)(\nu_B - \delta_B).$$

Thanks to Corollary 1.0.3, $\nu_{\ell+1}$ is a laminate. Moreover, it is clear that $\overline{\nu}_{\ell+1} = A$ and

$$\operatorname{supp}(\nu_{\ell+1}) \subset \bigcup_{b=0}^{\ell} A^{i+b}_{j+1} \cup B^{i+\ell}_{j+1} \cup \bigcup_{b=0}^{\ell} \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i+b}.$$

Now, observe that the matrices in $(A_{j+1}^{i+\ell}\cup B_{j+1}^{i+\ell}\cup\bigcup_{a=m_2+1}^{n-m_1-1}S_{j+1,i+\ell}^{a})\cap\operatorname{supp}(\nu_{\ell+1})$ are those in the support of ν_B for B in $B_{j+1}^{i+\ell-1}\cap\operatorname{supp}(\nu_\ell)$. Therefore, using $m_1'+m_2'\leqslant n'-1$ we have

$$\begin{aligned}
&\nu_{\ell+1}(A_{j+1}^{i+\ell}) = \sum_{B\in B_{j+1}^{i+\ell-1}}\nu_\ell(B)\nu_B(A_{j+1}^{i+\ell}) \leqslant \sum_{B\in B_{j+1}^{i+\ell-1}}\nu_\ell(B)(i+\ell)^{m_1'+m_2'-n'}\\
&= \nu_\ell(B_{j+1}^{i+\ell-1})(i+\ell)^{m_1'+m_2'-n'}\\
&\lesssim j^{2(m_1'+m_2'-n')}i^{m_1'}(i+\ell+1)^{m_2'-n'}\left(1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}\right)(i+\ell)^{m_1'+m_2'-n'}\\
&\leqslant j^{2(m_1'+m_2'-n')}i^{m_1'}(i+\ell+1)^{-m_1'-2}\left(1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}\right),
\end{aligned}$$

Therefore

$$\nu_{\ell+1}(A_{j+1}^{i+\ell})\lesssim j^{2(m_1'+m_2'-n')}\frac{i^{m_1'}}{(i+\ell)^{m_1'+2}}.$$

For the bound of $\nu_{\ell+1}(B_{j+1}^{i+\ell})$ we need the following estimate, in which we use $j\geqslant 2^nC$:

$$\begin{aligned}
&\left(1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}\right)\left(1+C\left(\frac{i+\ell+2}{i+\ell+1}\right)^{n'-m_2'}j^{-2}(i+\ell)^{-2}\right)\\
&\leqslant 1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}+2C2^{n-1}j^{-2}(i+\ell)^{-2}\leqslant 1+j^{-2}\sum_{k=1}^{\ell}2^nC(i+k)^{-2}.
\end{aligned}$$

Proceeding in the same way as in the bound of $\nu_{\ell+1}(A_{j+1}^{i+\ell})$, we obtain the following bound. Note that the constant corresponding to the symbol $\lesssim$ is the same for all $\ell=1,\dots,j-i+2$:

$$\begin{aligned}
&\nu_{\ell+1}(B_{j+1}^{i+\ell}) = \sum_{B\in B_{j+1}^{i+\ell-1}}\nu_\ell(B)\nu_B(B_{j+1}^{i+\ell}) \leqslant \sum_{B\in B_{j+1}^{i+\ell-1}}\nu_\ell(B)\left[\left(\frac{i+\ell+1}{i+\ell+2}\right)^{n'-m_2'}+C(i+\ell)^{-2}j^{-2}\right]\\
&= \nu_\ell(B_{j+1}^{i+\ell-1})\left[\left(\frac{i+\ell+1}{i+\ell+2}\right)^{n'-m_2'}+C(i+\ell)^{-2}j^{-2}\right]\\
&\lesssim j^{2(m_1'+m_2'-n')}i^{m_1'}(i+\ell+1)^{m_2'-n'}\left(1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}\right)\\
&\quad\times\left[\left(\frac{i+\ell+1}{i+\ell+2}\right)^{n'-m_2'}+C(i+\ell)^{-2}j^{-2}\right]\\
&\leqslant j^{2(m_1'+m_2'-n')}i^{m_1'}(i+\ell+2)^{m_2'-n'}\left(1+j^{-2}\sum_{k=1}^{\ell}2^nC(i+k)^{-2}\right).
\end{aligned}$$

Next we bound $\nu_{\ell+1}(S^a_{j+1,i+\ell})$ for $a\in\{m_2+1,\dots,n-m_1-1\}$:

$$\begin{aligned}\nu_{\ell+1}(S^a_{j+1,i+\ell}) &= \sum_{B\in B^{i+\ell-1}_{j+1}} \nu_\ell(B)\nu_B(S^a_{j+1,i+\ell}) \leqslant \sum_{B\in B^{i+\ell-1}_{j+1}} \nu_\ell(B)(i+\ell)^{m_2'-a}\\ &= \nu_\ell(B^{i+\ell-1}_{j+1})(i+\ell)^{m_2'-a}\\ &\lesssim j^{2(m_1'+m_2'-n')} i^{m_1'}(i+\ell+1)^{m_2'-n'}\left(1+j^{-2}\sum_{k=1}^{\ell-1}2^nC(i+k)^{-2}\right)(i+\ell)^{m_2'-a}\\ &\lesssim j^{2(m_1'+m_2'-n')} i^{m_1'}(i+\ell)^{2m_2'-a-n'}.\end{aligned}$$

For $b\in\{0,\dots,\ell\}$, we also have that

$$\nu_{\ell+1}|_{A^{i+b}_{j+1}} = \nu_\ell|_{A^{i+b}_{j+1}}$$

and

$$\nu_{\ell+1}|_{S^a_{j+1,i+b}} = \nu_\ell|_{S^a_{j+1,i+b}} \text{ for } a\in\{m_2+1,\dots,n-m_1-1\}.$$

Here, $|$ denotes the restriction of a measure. Therefore, $\nu_{\ell+1}$ satisfies 1)–7). Here ends the inductive construction of $\{\nu_\ell\}_{\ell=1}^{j-i+2}$.

Now, we define $\nu=\nu_{j-i+2}=\sum_{k=1}^N\lambda_k\delta_{M_\ell}$, so $\lambda_1=\nu_1(M_1)$ and we recall that

$$M_1\in A^i_{j+1}\cap\operatorname{supp}(\nu),\ |A-M_1|\leqslant j^{-2},\ |A^{-1}-M_1^{-1}|\lesssim 1 \text{ and } 1-\lambda_1\lesssim j^{-2}i^{-1}.$$

Finally we have to prove *h)* and *i)*. To show *h)* we need the following estimate of the distance between the matrices in the support of ν and A:

$$|A-M|\leqslant|A|+|M|\lesssim\begin{cases} i+b & \text{if } M\in A^{i+b}_{j+1}\setminus\{M_1\} \text{ for some } b\in\{0,\dots,j-i+1\},\\ j & \text{if } M\in B^{j+1}_{j+1},\\ i+b & \text{if } M\in S^a_{j+1,i+b} \text{ for some } b\in\{0,\dots,j-i+1\}.\end{cases}$$

Now we split the sum $\sum_{k=1}^N\lambda_k|A-M_k|$ over the different sets and we bound those sums:

$$\lambda_1|A-M_1|\lesssim j^{-2},$$

$$\sum_{k:M_k\in A^i_{j+1}\setminus\{M_1\}}\lambda_k|A-M_k|\lesssim\sum_{k:M_k\in A^i_{j+1}\setminus\{M_1\}}\lambda_k i\leqslant i(1-\lambda_1)\lesssim j^{-2}.$$

In the following estimate we use $\sum_{b=0}^\infty(i+b)^{-m_1-1}\lesssim i^{-m_1}$ and $m_1'+m_2'-n'\leqslant -1$:

$$\begin{aligned}\sum_{b=1}^{j-i+1}\sum_{k:M_k\in A^{i+b}_{j+1}}\lambda_k|A-M_k| &\lesssim \sum_{b=1}^{j-i+1}\sum_{k:M_k\in A^{i+b}_{j+1}}\lambda_k(i+b)=\sum_{b=1}^{j-i+1}\nu(A^{i+b}_{j+1})(i+b)\\ &\lesssim\sum_{b=1}^{j-i+1} j^{2(m_1'+m_2'-n')}\frac{i^{m_1'}}{(i+b)^{m_1'+1}}\lesssim j^{-2}.\end{aligned}$$

Using that $i \leqslant j$ and $m_1' + m_2' \leqslant n' - 1$, we have

$$\sum_{k:M_k \in B_{j+1}^{j+1}} \lambda_k |A - M_k| \lesssim \sum_{k:M_k \in B_{j+1}^{j+1}} \lambda_k j = \nu(B_{j+1}^{j+1}) j \lesssim j^{2m_1' + 3m_2' - 3n' + 1} i^{m_1'} \leqslant j^{-2}.$$

In the same way as before, we bound the sum over the sets $S_{j+1,i}^a$ and $S_{j+1,i}^{a+b}$; using that in this case $m_1' = m_1$ and $n' = n$, we get

$$\begin{aligned}
&\sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i}^a} \lambda_k |A - M_k| \lesssim \sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i}^a} \lambda_k i = \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i}^a) i \\
&\lesssim \sum_{a=m_2+1}^{n-m_1-1} j^{2(m_1'+a-n')} i^{m_1'+a-n'+1} \lesssim j^{-2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i+b}^a} \lambda_k |A - M_k| \lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i+b}^a} \lambda_k (i+b) \\
&= \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i+b}^a)(i+b) \lesssim \sum_{b=1}^{j-i+1} \sum_{a=m_2+1}^{n-m_1-1} j^{2(m_1'+m_2'-n')} i^{m_1'} (i+b)^{2m_2'-a-n'+1} \\
&\lesssim j^{-2} i^{m_1'} \sum_{b=1}^{j-i+1} (i+b)^{m_2'-n'} \leqslant j^{-2} i^{m_1'} \sum_{b=1}^{j-i+1} (i+b)^{-m_1'-1} \lesssim j^{-2}.
\end{aligned}$$

Therefore, putting together all the previous bounds we obtain

$$\sum_{k=1}^{N} \lambda_k |A - M_k| \lesssim j^{-2},$$

and, hence, *h)* is proved.

Now, to prove *i)* we need to bound the distance between the inverses. For $M \in \operatorname{supp}(\nu) \setminus \{M_1\}$ we have

$$|A^{-1} - M^{-1}| \leqslant |A^{-1}| + |M^{-1}| \lesssim j.$$

We also need the following bound of the determinants. Since $A \in A_j^i$ and $m_1 - n = m_1' - n'$ we have

$$\det A \gtrsim j^{m_1-n} i^{m_1'} = j^{m_1'-n'} i^{m_1'}.$$

Looking at the definition of the sets we also get

$$\det M \lesssim \begin{cases} j^{m_1'-n'}(i+b)^{m_1'} & \text{if } M \in A_{j+1}^{i+b} \text{ for some } b \in \{0, \dots, j-i+1\}, \\ j^{n'-2m_2'} & \text{if } M \in B_{j+1}^{j+1}, \\ j^{-a}(i+b)^{n'-a} & \text{if } M \in S_{j+1,i+b}^a \text{ for some } b \in \{0, \dots, j-i+1\}. \end{cases}$$

Hence we get

$$\frac{\det M}{\det A}|A^{-1}-M^{-1}| \lesssim \begin{cases} j(i+b)^{m_1'} i^{-m_1'} & \text{if } M \in A_{j+1}^{i+b} \setminus \{M_1\} \text{ for some} \\ & b \in \{0,\dots,j-i+1\}, \\ j^{2n'-2m_2'-m_1'+1} i^{-m_1'} & \text{if } M \in B_{j+1}^{j+1}, \\ j^{n'-m_1'-a+1}(i+b)^{n'-a} i^{-m_1'} & \text{if } M \in S_{j+1,i+b}^{a} \text{ for some} \\ & b \in \{0,\dots,j-i+1\}. \end{cases}$$

Now we split the sum $\sum_{k=1}^{N} \lambda_k \frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}|$ over the different sets and bound those sums:

$$\lambda_1 \frac{\det M_1}{\det A}|A^{-1}-M_1^{-1}| \lesssim \frac{\det M_1}{\det A} \lesssim j^{m_1'-n'} i^{m_1'} j^{n'-m_1'} i^{-m_1'} = 1,$$

$$\sum_{k:M_k \in A_{j+1}^{i} \setminus \{M_1\}} \lambda_k \frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}| \lesssim \sum_{k:M_k \in A_{j+1}^{i} \setminus \{M_1\}} \lambda_k j \leqslant j(1-\lambda_1) \lesssim j^{-1} i^{-1} \leqslant 1.$$

In the following estimate we use $\sum_{b=0}^{\infty}(i+b)^{-2} \lesssim i^{-1}$ and $m_1'+m_2'-n' \leqslant -1$:

$$\begin{aligned} &\sum_{b=1}^{j-i+1} \sum_{k:M_k \in A_{j+1}^{i+b}} \lambda_k \frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}| \lesssim \sum_{b=1}^{j-i+1} \sum_{k:M_k \in A_{j+1}^{i+b}} \lambda_k j \frac{(i+b)^{m_1'}}{i^{m_1'}} \\ &\leqslant \sum_{b=1}^{j-i+1} \nu(A_{j+1}^{i+b}) j \frac{(i+b)^{m_1'}}{i^{m_1'}} = \sum_{b=1}^{j-i+1} j^{2(m_1'+m_2'-n')+1} \frac{i^{m_1'}}{(i+b)^{m_1'+2}} \frac{(i+b)^{m_1'}}{i^{m_1'}} \lesssim j^{-1} i^{-1} \leqslant 1. \end{aligned}$$

Using that $i \leqslant j$ and $m_1'+m_2' \leqslant n'-1$, we have

$$\begin{aligned} &\sum_{k:M_k \in B_{j+1}^{j+1}} \lambda_k \frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}| \lesssim \sum_{k:M_k \in B_{j+1}^{j+1}} \lambda_k j^{2n'-2m_2'-m_1'+1} i^{-m_1'} \\ &= \nu(B_{j+1}^{j+1}) j^{2n'-2m_2'-m_1'+1} i^{-m_1'} \lesssim j^{2m_1'+3m_2'-3n'} i^{m_1'} j^{2n'-2m_2'-m_1'+1} i^{-m_1'} = j^{m_1'+m_2'-n'+1} \leqslant 1. \end{aligned}$$

In the same way as before we bound the sum over the sets $S_{j+1,i}^{a}$ and $S_{j+1,i}^{a+b}$; using that in this case we have $m_1' = m_1$ and $n' = n$, we get

$$\begin{aligned} &\sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i}^{a}} \lambda_k \frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}| \lesssim \sum_{a=m_2+1}^{n-m_1-1} \sum_{k:M_k \in S_{j+1,i}^{a}} \lambda_k j^{n'-m_1'-a+1} i^{n'-a-m_1'} \\ &= \sum_{a=m_2+1}^{n-m_1-1} \nu(S_{j+1,i}^{a}) j^{n'-m_1'-a+1} i^{n'-a-m_1'} \lesssim \sum_{a=m_2+1}^{n-m_1-1} j^{2(m_1'+a-n')} i^{m_1'+a-n'} j^{n'-m_1'-a+1} i^{n'-a-m_1'} \\ &= \sum_{a=m_2+1}^{n-m_1-1} j^{m_1'-n'+a+1} \lesssim 1 \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{b=1}^{j-i+1}\sum_{a=m_2+1}^{n-m_1-1}\sum_{k:M_k\in S^a_{j+1,i+b}}\lambda_k\frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}|\\
&\lesssim \sum_{b=1}^{j-i+1}\sum_{a=m_2+1}^{n-m_1-1}\sum_{k:M_k\in S^a_{j+1,i+b}}\lambda_k j^{n'-m_1'-a+1}(i+b)^{n'-a}i^{-m_1'}\\
&= \sum_{b=1}^{j-i+1}\sum_{a=m_2+1}^{n-m_1-1}\nu(S^a_{j+1,i+b})j^{n'-m_1'-a+1}(i+b)^{n'-a}i^{-m_1'}\\
&\lesssim \sum_{b=1}^{j-i+1}\sum_{a=m_2+1}^{n-m_1-1}j^{m_1'+2m_2'-n'-a+1}(i+b)^{2m_2'-2a}\lesssim j^{-1}\sum_{b=1}^{j-i+1}(i+b)^{-2}\lesssim 1.
\end{aligned}
$$

Therefore, we get

$$\sum_{k=1}^{N}\lambda_k\frac{\det M_k}{\det A}|A^{-1}-M_k^{-1}|\lesssim 1.$$

The proof is finished. □

The proofs of the next lemmas are analogous to the proof of Lemma 2.3.9, but instead of using Lemma 2.3.6 in the first step and Lemma 2.3.7 in the induction, we use Lemma 2.3.7 in the first step and Lemma 2.3.6 in the induction for Lemma 2.3.10. For Lemmas 2.3.11 and 2.3.12 we use Lemma 2.3.8 in the first step, using in the induction Lemma 2.3.6 for Lemma 2.3.12 and Lemma 2.3.7 for Lemma 2.3.11.

Lemma 2.3.10. *Let $i,j\in\mathbb{N}$, $i\leqslant j$, and $A\in B_i^j$. Then, there exists a laminate $\nu=\sum_{k=1}^N\lambda_k\delta_{M_k}$ such that*

a) $\overline{\nu}=A$,

b) $\operatorname{supp}(\nu)\subset A^{j+1}_{j+1}\cup\bigcup_{b=0}^{j-i+1}B^{j+1}_{i+b}\cup\bigcup_{b=0}^{j-i+1}\bigcup_{a=m_2+1}^{n-m_1-1}S^a_{i+b,j+1}$,

c) $\nu(A^{j+1}_{j+1})\lesssim j^{m_1'+m_2'-n'}$,

d) $\nu(B_i^{j+1})-\left(\frac{j+2}{j+3}\right)^{n'-m_2'}\lesssim (ij)^{-2}$,

e) $\nu(B^{j+1}_{i+b})\lesssim\big((i+b)(j+3)\big)^{2(m_1'+m_2'-n')}$, *for* $b\in\{1,\dots,j-i+1\}$,

f) $\nu(S^a_{i,j+1})\lesssim j^{m_2'-a}$ *for* $a\in\{m_2+1,\dots,n-m_1-1\}$,

g) $\nu(S^a_{i+b,j+1})\lesssim j^{m_1'+m_2'-n'}\big((i+b-1)^2(j+1)\big)^{m_1'+a-n'}$ *for* $a\in\{m_2+1,\dots,n-m_1-1\}$ *and* $b\in\{1,\dots,j-i+1\}$,

h) $\sum_{k=1}^N\lambda_k|A-M_k|\lesssim 1$,

i) $\frac{1}{\det(A)}\sum_{k=1}^N\lambda_k\det(M_k)|A^{-1}-M_k^{-1}|\lesssim j^{-2}$.

Lemma 2.3.11. *Let $a_0 \in \{m_2+1,\dots,n-m_1-1\}$, $i,j \in \mathbb{N}$, $i \leqslant j$, and $A \in S^{a_0}_{j,i}$. Then there exists a laminate $\nu = \sum_{k=1}^N \lambda_k \delta_{M_k}$ such that*

a) $\overline{\nu} = A$,

b) $\operatorname{supp}(\nu) \subset \bigcup_{b=1}^{j-i+1} A^{i+b}_{j+1} \cup B^{j+1}_{j+1} \cup \bigcup_{b=1}^{j-i+1} \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j+1,i+b}$,

c) $\nu(A^{i+1}_{j+1}) \lesssim i^{a_0+m_1-n}$,

d) $\nu(A^{i+1+b}_{j+1}) \lesssim j^{2(m_2'-a_0)} \frac{i^{n'-a_0}}{(i+b)^{m_1'+2}}$, *for* $b \in \{1,\dots,j-i\}$,

e) $\nu(B^{j+1}_{j+1}) \lesssim j^{3m_2'-n'-2a_0} i^{n'-a_0}$,

f) $\nu(S^a_{j+1,i+1}) \lesssim (j^2 i)^{a-a_0}$ *if* $a \in \{m_2+1,\dots,a_0-1\}$,

g) $\nu(S^{a_0}_{j+1,i+1}) - \left(\frac{i+2}{i+3}\right)^{n-a_0} \lesssim (ji)^{-2}$,

h) $\nu(S^a_{j+1,i+1}) \lesssim i^{a_0-a}$ *if* $a \in \{a_0+1,\dots,n-m_1-1\}$,

i) $\nu(S^a_{j+1,i+1+b}) \lesssim j^{2(m_2'-a_0)} i^{n'-a_0} (i+b)^{2m_2'-a-n'}$ *for* $a \in \{m_2+1,\dots,n-m_1-1\}$ *and* $b \in \{1,\dots,j-i\}$,

j) $\sum_{k=1}^N \lambda_k |A-M_k| \lesssim 1$,

k) $\frac{1}{\det(A)} \sum_{k=1}^N \lambda_k \det(M_k) |A^{-1} - M_k^{-1}| \lesssim 1$.

Lemma 2.3.12. *Let $a_0 \in \{m_2+1,\dots,n-m_1-1\}$, $i,j \in \mathbb{N}$, $i \leqslant j$, and $A \in S^{a_0}_{i,j}$. Then there exists a laminate $\nu = \sum_{k=1}^N \lambda_k \delta_{M_k}$ such that*

a) $\overline{\nu} = A$,

b) $\operatorname{supp}(\nu) \subset A^{j+1}_{j+1} \cup \bigcup_{b=1}^{j-i+1} B^{j+1}_{i+b} \cup \bigcup_{b=1}^{j-i+1} \bigcup_{a=m_2+1}^{n-m_1-1} S^a_{i+b,j+1}$,

c) $\nu(A^{j+1}_{j+1}) \lesssim j^{a_0+m_1'-n'}$,

d) $\nu(B^{j+1}_{i+1}) \lesssim (i^2 j)^{m_2'-a_0}$,

e) $\nu(B^{j+1}_{i+1+b}) \lesssim (i+b)^{2(m_1'+m_2'-n')} j^{2m_1'+m_2'-2n'+a_0}$, *for* $b \in \{1,\dots,j-i\}$,

f) $\nu(S^a_{i+1,j+1}) \lesssim (i^2 j)^{a-a_0}$ *if* $a \in \{m_2+1,\dots,a_0-1\}$,

g) $\nu(S^{a_0}_{i+1,j+1}) - \left(\frac{j+2}{j+3}\right)^{n-a_0} \lesssim (ji)^{-2}$,

h) $\nu(S^a_{i+1,j+1}) \lesssim j^{a_0-a}$ *if* $a \in \{a_0+1,\dots,n-m_1-1\}$,

i) $\nu(S^a_{i+1+b,j+1}) \lesssim j^{2m_1'-2n'+a+a_0} (i+b)^{2(m_1'+a-n')}$ *for* $a \in \{m_2+1,\dots,n-m_1-1\}$ *and* $b \in \{1,\dots,j-i\}$, *j)* $\sum_{k=1}^{N} \lambda_k |A - M_k| \lesssim 1$,

k) $\frac{1}{\det(A)} \sum_{k=1}^{N} \lambda_k \det(M_k)|A^{-1} - M_k^{-1}| \lesssim 1$.

We construct now two families of uniformly bounded constants that will be used in the proof of Theorem 2.3.1.

Lemma 2.3.13. *Let* $\varepsilon' > 0$ *and* $\tilde{C} > 1$. *Define the sequences* $\{C^1_{j,i}\}_{\substack{j\in\mathbb{N}\\ i=1,\dots,j}}$, $\{C^2_{j,i}\}_{\substack{j\in\mathbb{N}\\ i=0,\dots,j}}$ *and* $\{M_j\}_{j\in\mathbb{N}}$ *as follows:*

a) $C^1_{1,1} = 4^n$, $C^2_{1,1} = 4^n$, $C^2_{j,0} = 0$ *for* $j \in \mathbb{N}$.

Given $j \in \mathbb{N}$, *assuming* $C^1_{j,i}$ *and* $C^2_{j,i}$ *have been defined for all* $i \in \{1,\dots,j\}$, *set*

b) $M_j = \max_{i=1,\dots,j} C^1_{j,i}$ *and* $M_0 = 4^n$,

c) $C^1_{j+1,i} = C^1_{j,i} + j^{-2}\tilde{C}M_j + \tilde{C}C^2_{j,i-1}(j+2-i)^{-2} + j^{-2}\tilde{C}\sum_{l=1}^{i-2} C^2_{j,l}(j+1-l)^{-2}$, *for* $i = 1,\dots,j$,

d) $C^1_{j+1,j+1} = \tilde{C}\left(M_j + C^2_{j,j} + \sum_{l=1}^{j} C^2_{j,l}(j+1-l)^{-2}\right) j^{-\varepsilon'}$,

e) $C^2_{j+1,i} = C^2_{j,i-1}\left(1 + \tilde{C}i^{-2}\right) + \tilde{C}M_j i^{-2+\varepsilon'} + j^{-2}i^{-2}\tilde{C}\sum_{l=1}^{i-2} C^2_{j,l}$, *for* $i = 1,\dots,j$,

f) $C^2_{j+1,j+1} = C^2_{j,j}\left(1 + \tilde{C}j^{-2}\right) + \tilde{C}j^{-4}\left(M_j + \sum_{l=1}^{j-1} C^2_{j,l}\right)$,

Then

$$\sup_{j\in\mathbb{N}} M_j < \infty \text{ and } \sup_{\substack{j\in\mathbb{N}\\ i=0,\dots,j}} C^2_{j,i} < \infty.$$

Proof. Clearly $M_j \leqslant M_{j+1}$ for $j \in \mathbb{N}$. Define $K_j = \prod_{\ell=1}^{j}\left(1 + 8\tilde{C}\ell^{-2+\varepsilon'}\right)$. We will prove by induction in j that for all $j \in \mathbb{N}$ and $i = 1,\dots,j$ we have $C^2_{j,i} \leqslant K_i M_{j-1}$. For $j = 1$ is obvious, so suppose that it is true for j and we will prove it for $j+1$. Since M_j and K_j are increasing with j we get

$$C^2_{j+1,j+1} \leqslant K_j M_{j-1}(1 + \tilde{C}j^{-2}) + \tilde{C}j^{-3}\left(M_j + K_{j-1}M_{j-1}\right) \leqslant K_j M_j (1 + 8\tilde{C}(j+1)^{-2+\varepsilon'}) = K_{j+1}M_j$$

and

$$\begin{aligned} C^2_{j+1,i} &\leqslant K_{i-1}M_{j-1}(1 + \tilde{C}i^{-2}) + \tilde{C}M_j i^{-2+\varepsilon'} + j^{-1}i^{-2}\tilde{C}K_{i-1}M_{j-1} \\ &\leqslant K_{i-1}M_j(1 + 8\tilde{C}i^{-2+\varepsilon'}) = K_i M_j. \end{aligned}$$

Therefore, there exists a constant $K > 0$ such that $C^2_{j+1,i} \leqslant KM_j$ for all $j \in \mathbb{N}$ and $i = 1,\dots,j+1$. Hence, it is enough proof that the supremum of $\{M_j\}_{j\in\mathbb{N}}$ is finite.

From *d)* we get

$$C^1_{j+1,j+1} \lesssim M_j j^{-\varepsilon'}. \tag{2.3.20}$$

On the other hand, proceeding as before we can prove that

$$C^2_{j+1,i} \lesssim \sum_{\ell=1}^{i} M_{j+\ell-i}\ell^{-2+\varepsilon'}. \tag{2.3.21}$$

By *c)* we obtain by induction on $j \geqslant i-1$ that

$$C^1_{j+1,i} \leqslant C^1_{i,i} + \sum_{\ell=i}^{j} 3\tilde{C}K\left(M_\ell \ell^{-2} + (\ell+2-i)^{-2}C^2_{\ell,i-1}\right). \tag{2.3.22}$$

For $r \geqslant 1$ we use the numerical inequality

$$-s^2+(r-2)s+2r \geqslant \begin{cases} \frac{r(s+1)}{2} & \text{for } 0 \leqslant s \leqslant \frac{r}{2}, \\ \frac{r(r-s)}{2} & \text{for } \frac{r}{2} \leqslant s \leqslant r-1 \end{cases}$$

to get

$$\begin{aligned} &\sum_{\ell=i}^{j}\left[(\ell+2-i)^{-2}\sum_{k=1}^{i-1} M_{\ell+k-i}k^{-2+\varepsilon'}\right] \leqslant \sum_{r=1}^{j-1} M_r \sum_{s=0}^{r-1}(s+2)^{-2}(r-s)^{-2+\varepsilon'} \\ &\leqslant \sum_{r=1}^{j-1} M_r \sum_{s=0}^{r-1}(-s^2+(r-2)s+2r)^{-2+\varepsilon'} \lesssim \sum_{\ell=1}^{j-1} M_\ell \ell^{-2+\varepsilon'}. \end{aligned} \tag{2.3.23}$$

Now, we use (2.3.21), (2.3.22) and (2.3.23) to get that there exists a constant C' depending only on n such that

$$C^1_{j+1,i} \leqslant C^1_{i,i} + C'\sum_{\ell=1}^{j} M_\ell \ell^{-2+\varepsilon'}.$$

Let $j_{\varepsilon'}$ be such that for all $j \geqslant j_{\varepsilon'}$

$$C^1_{j+1,j+1} \leqslant M_j,$$

which is possible thanks to (2.3.20). Define the family of constants $\{C^3_{j,i}\}_{\substack{j\in\mathbb{N}\\ i=1,\dots,j}}$ as follows

$$C^3_{j,i} = \begin{cases} C^1_{j,i} & \text{if } j \leqslant j_{\varepsilon'} \text{ or } i=j, \\ C^1_{i,i} + C'\sum_{\ell=1}^{j-1} M_\ell \ell^{-2+\varepsilon'} & \text{if } j > \max\{i, j_{\varepsilon'}\}. \end{cases}$$

Hence, $C^1_{j,i} \leqslant C^3_{j,i}$ for all $j \in \mathbb{N}$ and $i = 1,\dots,j$. Define $M'_j = \max_{i=1,\dots,j} C^3_{j,i}$ and let $i_j \in \{1,\dots,j\}$ be such that $M'_j = C^3_{j,i_j}$. First, we note that $M'_j < M'_{j+1}$ for all $j > j_{\varepsilon'}$. Fix $j > j_{\varepsilon'}$; as

$$C^3_{j+1,j+1} = C^1_{j+1,j+1} \leqslant M_j \leqslant M'_j \text{ and } M'_j < M'_{j+1},$$

it is clear that $i_{j+1} \leqslant j$. We also see that

$$C^3_{j+1,i_j} = C^3_{j,i_j} + C'M_j j^{-2+\varepsilon'} = M'_j + C'M_j j^{-2+\varepsilon'},$$

and for all $i \leqslant j$ we have

$$C^3_{j+1,i} = C^3_{j,i} + C'M_j j^{-2+\varepsilon'}.$$

Therefore $M'_{j+1} = C^3_{j+1,i_j}$, so we can take $i_{j+1} = i_j$. By induction, we can take $i_j = i_{j_{\varepsilon'}+1}$ for all $j > j_{\varepsilon'}$, and $M'_j = C^3_{j,i_{j_{\varepsilon'}+1}}$. Hence, there exists $C > 0$ such that

$$M'_{j+1} \leqslant C \sum_{\ell=1}^{j} M_\ell \ell^{-2+\varepsilon'} \leqslant C \sum_{\ell=1}^{j} M'_\ell \ell^{-2+\varepsilon'}.$$

Define for $j \leqslant j_{\varepsilon'} + 1$, $\tilde{M}_j = M'_j$ and for $j \geqslant j_{\varepsilon'} + 1$

$$\tilde{M}_{j+1} = C \sum_{\ell=1}^{j} \tilde{M}_\ell \ell^{-2+\varepsilon'},$$

so $M'_j \leqslant \tilde{M}_j$ for all $j \in \mathbb{N}$. Now, we observe that for $j \geqslant j_{\varepsilon'} + 2$ we have

$$\tilde{M}_{j+1} = \tilde{M}_j (1 + C j^{-2+\varepsilon'}).$$

Therefore

$$\sup_{j\in\mathbb{N}} M_j \leqslant \sup_{j\in\mathbb{N}} M'_j \leqslant \sup_{j\in\mathbb{N}} \tilde{M}_j < \infty$$

and the proof is completed. □

Finally, we combine all the previous results to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. In this proof, expressions like $\{x \in \Omega : f(x) \in A\}$ will be abbreviated as $\{f(x) \in A\}$. Given $\varepsilon' > 0$ small enough to have $\sum_{k=1}^{\infty} k^{-2+\varepsilon'} < 2$, we will construct a sequence $\{f_j\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega,\Omega)$ of piecewise affine Lipschitz homeomorphisms such that $f_0 = \mathrm{id}$ and, when we take $\{C^1_{j,i}\}_{\substack{j\in\mathbb{N}\\ i=1,\dots,j}}$, $\{C^2_{j,i}\}_{\substack{j\in\mathbb{N}\\ i=0,\dots,j}}$ the families of constants in Lemma 2.3.13 and we denote $\Omega^j_S = \{Df_j(x) \in S\}$ for each $S \subset \Gamma_+$, we have

i) $f_j = \mathrm{id}$ on $\partial\Omega$,

ii) $\|f_j - f_{j-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-j}\varepsilon$ and $\|f_j^{-1} - f_{j-1}^{-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-j}\varepsilon$,

iii) $Df_j(x) \in \bigcup_{i=1}^{j} \bigcup_{a=m_2+1}^{n-m_1-1} \left(A^i_j \cup B^j_i \cup S^a_{j,i} \cup S^a_{i,j} \right)$,

iv) $\int_\Omega |Df_j(x) - Df_{j-1}(x)|\,dx \lesssim j^{-2}|\Omega|$,

v) $\int_\Omega |Df_j^{-1}(y) - Df_{j-1}^{-1}(y)|\,dy \lesssim j^{-2}|\Omega|$,

vi) $\frac{|\Omega^j_{A^i_j}|}{|\Omega|} \leqslant C^1_{j,i}\, i^{-m'_1-2+\varepsilon'}$ for $i = 1,\dots,j$,

vii) $\frac{|\Omega^j_{B^j_i}|}{|\Omega|} \leqslant C^1_{j,i}\, i^{-2+\varepsilon'}(j+2)^{m'_2-n'}$ for $i = 1,\dots,j$,

viii) $\frac{|\Omega^j_{S^a_{j,i}}|}{|\Omega|} \leqslant C^2_{j,i}(i+2)^{a-n'}(j+1-i)^{-2}$ for $i = 1,\dots,j$ and $a \in \{m_2+1,\dots,n-m_1-1\}$,

ix) $\frac{|\Omega^j_{S^a_{i,j}}|}{|\Omega|} \leqslant C^2_{j,i}(j+2)^{a-n'}(j+1-i)^{-2}$ for $i=1,\dots,j$ and $a \in \{m_2+1,\dots,n-m_1-1\}$.

Once constructed such sequence $\{f_j\}$, we have that it converges in the C^α and in the $W^{1,1}$ norm to a bi-Sobolev homeomorphism $f:\Omega\to\Omega$; see, if necessary, the proof of Theorem 2.2.1 for the details of the limit passage. Moreover, it is immediate from i), ii), that f satisfies *i)* and *ii)* of Theorem 2.3.1. Recall that the bounds of $S^a_{i,k}$ only have sense if $m_1+m_2 \leqslant n-1$; otherwise, these sets are empty. From iii), vii), viii) and ix) we obtain

$$1-\frac{|\Omega^j_{\bigcup_{i=1}^j A^i_j}|}{|\Omega|} \lesssim \sum_{i=1}^{j}\left[i^{-2+\varepsilon'}j^{m_2'-n'}+(j+1-i)^{-2}\left(\sum_{a=m_2+1}^{n-m_1-1} i^{a-n'}+j^{a-n'}\right)\right] \lesssim j^{-1}. \tag{2.3.24}$$

For a subsequence, $Df_j \to Df$ a.e., so, thanks to the continuity of the singular values and using that Γ_+ is closed we obtain that $Df\in\Gamma_+$ and, thanks to (2.3.24), we also get that $\operatorname{rank}\big(Df\big)=m_1$ a.e. in Ω. On the other hand,

$$\begin{aligned}
1-\frac{\left|\left\{Df_j^{-1}(y)\in\bigcup_{i=1}^j\left(B^j_i\right)^{-1}\right\}\right|}{|\Omega|} &= \frac{\left|\left\{Df_j^{-1}(y)\in\bigcup_{i=1}^j\bigcup_{a=m_2+1}^{n-m_1-1}\left(A^i_j\cup S^a_{i,j}\cup S^a_{j,i}\right)^{-1}\right\}\right|}{|\Omega|}\\
&=\frac{1}{|\Omega|}\left|f_j\left(\left\{Df_j(x)\in\bigcup_{i=1}^j\bigcup_{a=m_2+1}^{n-m_1-1}A^i_j\cup S^a_{i,j}\cup S^a_{j,i}\right\}\right)\right|\\
&=\frac{1}{|\Omega|}\int_{\left\{Df_j(x)\in\bigcup_{i=1}^j\bigcup_{a=m_2+1}^{n-m_1-1}A^i_j\cup S^a_{i,j}\cup S^a_{j,i}\right\}}\det Df_j(x)\,dx.
\end{aligned}$$

Now, we split the integral over the different sets and we use the control that we have over the determinant in the different sets (the second part with $M\in B^j_i$ will be used later):

$$\det M \lesssim \begin{cases} j^{m_1'-n'}i^{m_1'} & \text{if } M\in A^i_j,\\ j^{n'-m_2'}i^{-m_2'} & \text{if } M\in B^j_i,\\ k^{-a}i^{n'-a} & \text{if } M\in S^a_{k,i}.\end{cases} \tag{2.3.25}$$

Therefore using vi), viii) and ix) we get

$$\begin{aligned}
&1-\frac{\left|\left\{Df_j^{-1}(x)\in\bigcup_{i=1}^j\left(B^j_i\right)^{-1}\right\}\right|}{|\Omega|}\\
&\lesssim \sum_{i=1}^{j}\left[i^{-m_1'-2+\varepsilon'}j^{m_1'-n'}i^{m_1'}+(j+1-i)^{-2}\sum_{a=m_2+1}^{n-m_1-1}\left(i^{a-n'}j^{-a}i^{n'-a}+j^{a-n'}i^{-a}j^{n'-a}\right)\right]\lesssim j^{-1}.
\end{aligned}$$

The same argument as before shows that $\operatorname{rank}(Df^{-1}(y))=m_2$ a.e. $y\in\Omega$. Hence, *iii)* of Theorem 2.3.1 is proved.

Let $\varepsilon, t>0$ and pick $j>t$; then using that

$$|M| \leqslant \begin{cases} i+2 & \text{if } M\in A^i_j,\\ j+2 & \text{if } M\in B^j_i,\\ i+2 & \text{if } M\in S^a_{k,i},\end{cases}$$

we have

$$\frac{|\{|Df_j(x)|>t\}|}{|\Omega|} \leqslant \sum_{i=\lfloor t\rfloor-1}^{j}\left[\frac{|\Omega^j_{A^i_j}|}{|\Omega|}+\sum_{a=m_2+1}^{n-m_1-1}\frac{|\Omega^j_{S^a_{j,i}}|}{|\Omega|}\right]+\sum_{i=1}^{j}\left[\frac{|\Omega^j_{B^j_i}|}{|\Omega|}+\sum_{a=m_2+1}^{n-m_1-1}\frac{|\Omega^j_{S^a_{i,j}}|}{|\Omega|}\right]$$
$$\lesssim \sum_{i=\lfloor t\rfloor-1}^{j}\left[i^{-m_1'-2+\varepsilon'}+\sum_{a=m_2+1}^{n-m_1-1} i^{a-n'}(j+1-i)^{-2}\right]$$
$$+\sum_{i=1}^{j}\left[i^{-2+\varepsilon'}j^{m_2'-n'}+\sum_{a=m_2+1}^{n-m_1-1} j^{a-n'}(j+1-i)^{-2}\right].$$

Hence, using $\sum_{i=\lfloor t\rfloor-1}^{j} i^{-m_1'-1}(j+1-i)^{-2}\lesssim t^{-m_1'-1}\leqslant t^{-m_1'-1+\varepsilon'}$ and $m_1'+m_2'\leqslant n'-1$ we get

$$\frac{|\{|Df_j(x)|>t\}|}{|\Omega|}\lesssim t^{-m_1'-1+\varepsilon'},$$

and, hence, since we have proved the bound for all $\varepsilon'>0$ we have $f\in W^{1,p}(\Omega,\mathbb{R}^n)$ for all $p<m_1'+1$.

Next, for the inverse we will use the bounds

$$|M^{-1}|\leqslant\begin{cases} j+2 & \text{if } M\in A^i_j,\\ i+2 & \text{if } M\in B^j_i,\\ k+2 & \text{if } M\in S^a_{k,i}.\end{cases}$$

Therefore we get

$$\frac{|\{|Df_j^{-1}(y)|>t\}|}{|\Omega|}=\frac{|f_j\left(\{|Df_j(x)^{-1}|>t\}\right)|}{|\Omega|}$$
$$\leqslant\sum_{i=1}^{j}\left[\frac{|f_j(\Omega^j_{A^i_j})|}{|\Omega|}+\sum_{a=m_2+1}^{n-m_1-1}\frac{|f_j(\Omega^j_{S^a_{j,i}})|}{|\Omega|}\right]+\sum_{i=\lfloor t\rfloor-1}^{j}\left[\frac{|f_j(\Omega^j_{B^j_i})|}{|\Omega|}+\sum_{a=m_2+1}^{n-m_1-1}\frac{|f_j(\Omega^j_{S^a_{i,j}})|}{|\Omega|}\right].$$

Therefore, using (2.3.25) we get

$$\frac{|\{|Df_j^{-1}(y)|>t\}|}{|\Omega|}\lesssim\sum_{i=1}^{j}\left[j^{m_1'-n'}i^{m_1'}i^{-m_1'-2+\varepsilon'}+\sum_{a=m_2+1}^{n-m_1-1}j^{-a}i^{n'-a}i^{a-n'}(j+1-i)^{-2}\right]$$
$$+\sum_{i=\lfloor t\rfloor-1}^{j}\left[j^{n'-m_2'}i^{-m_2'}i^{-2+\varepsilon'}j^{m_2'-n'}+\sum_{a=m_2+1}^{n-m_1-1}i^{-a}j^{n'-a}j^{a-n'}(j+1-i)^{-2}\right]\lesssim t^{-m_2'-1+\varepsilon'}.$$

So, we have $f^{-1}\in W^{1,q}(\Omega,\mathbb{R}^n)$ for all $q<m_2+1$, and therefore, part *iv)* of Theorem 2.3.1 is proved.

Hence, to prove the theorem it is enough to construct the sequence $\{f_j\}$. Let $f_0=\mathrm{id}$ and proceeding as in Lemma 2.3.6 we construct a laminate ν_1 such that $\overline{\nu}_1=I$ and

$$\operatorname{supp}(\nu_1)\subset A^1_1\cup B^1_1\cup\bigcup_{a=m_2+1}^{n-m_1-1}S^a_{1,1}.$$

Now apply Proposition 1.3.5 with δ small enough to have $Df_1(x) \in A_1^1 \cup B_1^1 \cup \bigcup_{a=m_2+1}^{n-m_1-1} S_{1,1}^a$ for almost every $x \in \Omega$; this is possible because Proposition 1.3.5 gives us that $Df_1(x) \in \Gamma_+$ a.e. $x \in \Omega$; and, for $i, k \in \mathbb{N}$, $a \in \{m_2+1, \dots, n-m_1-1\}$, the sets $A_i^k, B_i^k, S_{i,k}^a$ are open in Γ_+. We do now the inductive step: suppose that we have constructed f_j. As f_j is piecewise affine, there exist families $\{A_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$, $\{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{\omega_k\}_{k\in\mathbb{N}}$ of disjoint sets such that $|\Omega \setminus (\bigcup_{k\in\mathbb{N}} \omega_k)| = 0$ and

$$f_j(x) = A_k x + b_k \text{ in } \omega_k.$$

For each k, let ν_k be the laminate given by Lemma 2.3.9 if $A_k \in \bigcup_{i=1}^j A_j^i$, the one given by Lemma 2.3.10 if $A_k \in \bigcup_{i=1}^j B_i^j$, the one given by Lemma 2.3.11 if $A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a$ and the one given by Lemma 2.3.12 if $A_k \in \bigcup_{i=1}^j \bigcup_{a=m_2+1}^{n-m_1-1} S_{i,j}^a$. For each k, let g_k be the homeomorphism given by Proposition 1.3.5 corresponding to ν_k that is equal to $A_k x + b_k$ on the border of ω_k, with $\delta_k > 0$ being as small as we need in the rest of the proof. First let δ_k be small enough to get

$$Dg_k(x) \in \bigcup_{i=1}^{j+1} \bigcup_{a=m_2+1}^{n-m_1-1} \left(A_{j+1}^i \cup B_i^{j+1} \cup S_{j+1,i}^a \cup S_{i,j+1}^a \right) \text{ for almost every } x \in \omega_k,$$

and such that for all $k \in \mathbb{N}$ we have $\delta_k < 2^{-j-1}\varepsilon$. Then, we define

$$f_{j+1} = \begin{cases} g_k & \text{in } \omega_k \text{ for some } k \in \mathbb{N}, \\ f_j & \text{in } \Omega \setminus \bigcup_{k\in\mathbb{N}} \omega_k. \end{cases}$$

It is obvious that f_{j+1} satisfies i) and iii); using Lemma 1.2.2 we see that it is a homeomorphism, and by Proposition 1.3.5 and Lemma 1.2.2 we have ii). Now we prove iv) and v). Choose δ_k such that, if $\nu_k = \sum_{\ell=1}^{N_k} \lambda_{k,\ell} \delta_{M_{k,\ell}}$,

$$\delta_k < \min_{\ell \in \{1,\dots,N_k\}} |A_k - M_{k,\ell}|, \tag{2.3.26}$$

and for $\ell = 1, \dots, N_k$,

$$|Dg_k^{-1}(y) - M_{k,\ell}^{-1}| < |A_k^{-1} - M_{k,\ell}^{-1}| \quad \text{in } g_k(\{x \in \omega_k : |Dg_k(x) - M_{k,\ell}| < \delta_k\}), \tag{2.3.27}$$

which is possible thanks to the continuity of the operator $A \to A^{-1}$ in the set of invertible matrices.

Denote by $\omega_{k,\ell}$ the set $\{x \in \omega_k : |Dg_k(x) - M_{k,\ell}| < \delta_k\}$. Recall, from Proposition 1.3.5 *(c)*, that $|\omega_{k,\ell}| = \lambda_{k,\ell} |\omega_k|$. Then, using part *h)* of Lemma 2.3.9, part *(c)* of Proposition 1.3.5 and (2.3.26), we have that for those $k \in \mathbb{N}$ such that $A_k \in \bigcup_{i=1}^j A_j^i$,

$$\begin{aligned} \int_{\omega_k} |A_k - Dg_k(x)| dx &\leq \sum_{\ell=1}^{N_k} \int_{\omega_{k,\ell}} \left(|A_k - M_{k,\ell}| + |M_{k,\ell} - Dg_k(x)| \right) dx \\ &\lesssim \sum_{\ell=1}^{N_k} \int_{\omega_{k,\ell}} |A_k - M_{k,\ell}| dx = \sum_{\ell=1}^{N_k} \lambda_{k,\ell} |A_k - M_{k,\ell}| |\omega_k| \lesssim j^{-2} |\omega_k|, \end{aligned}$$

and, also using part *i)* of Lemma 2.3.9, (2.3.27) and that

$$|g_k(\omega_{k,\ell})| \lesssim \det M_{k,\ell} |\omega_{k,\ell}| = \det M_{k,\ell} \lambda_{k,\ell} |\omega_k|$$

we get

$$\int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)|dy \leq \sum_{\ell=1}^{N_k} \int_{g_k(\omega_{k,\ell})} \left(|A_k^{-1} - M_{k,\ell}^{-1}| + |M_{k,\ell}^{-1} - Dg_k^{-1}(y)|\right) dy$$
$$\lesssim \sum_{\ell=1}^{N_k} \int_{g_k(\omega_{k,\ell})} |A_k^{-1} - M_{k,\ell}^{-1}|dy \lesssim \sum_{\ell=1}^{N_k} \lambda_{k,\ell} \det M_{k,\ell} |\omega_k||A_k^{-1} - M_{k,\ell}^{-1}| \lesssim \det A_k|\omega_k| = |g_k(\omega_k)|.$$

Proceeding in the same way as before, using Lemma 2.3.10 instead of Lemma 2.3.9, we obtain that for those $k \in \mathbb{N}$ such that $A_k \in \bigcup_{i=1}^{j} B_i^j$ we have

$$\int_{\omega_k} |A_k - Dg_k(x)|dx \lesssim |\omega_k|$$

and

$$\int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)|dy \lesssim j^{-2}|g_k(\omega_k)|.$$

Similarly, using Lemmas 2.3.11 and 2.3.12, we get that for $k \in \mathbb{N}$ such that $A_k \in \bigcup_{i=1}^{j} \bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a \cup S_{i,j}^a$ we have

$$\int_{\omega_k} |A_k - Dg_k(x)|dx \lesssim |\omega_k|$$

and

$$\int_{g_k(\omega_k)} |A_k^{-1} - Dg_k^{-1}(y)|dy \lesssim |g_k(\omega_k)|.$$

Now we combine the last equations, and we use that $Df_{j+1}(x) = Dg_k(x)$ and $Df_j(x) = A_k$ for x in ω_k, to prove iv) and v):

$$\int_{\Omega} |Df_{j+1} - Df_j|dx = \sum_{k\in\mathbb{N}} \int_{\omega_k} |Dg_k - A_k|dx$$
$$\lesssim \sum_{\substack{k\in\mathbb{N} \\ A_k\in\bigcup_{i=1}^{j} A_j^i}} j^{-2}|\omega_k| + \sum_{\substack{k\in\mathbb{N} \\ A_k\in\bigcup_{i=1}^{j} B_i^j}} |\omega_k| + \sum_{\substack{k\in\mathbb{N} \\ A_k\in\bigcup_{i=1}^{j}\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a\cup S_{i,j}^a}} |\omega_k|$$
$$= j^{-2}|\Omega^j_{\bigcup_{i=1}^{j} A_j^i}| + |\Omega^j_{\bigcup_{i=1}^{j} B_i^j}| + |\Omega^j_{\bigcup_{i=1}^{j}\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a\cup S_{i,j}^a}|.$$

Now, we use vi)–ix), that $m_2' - n' \leq -2$ and we recall that in the bounds for the sets $S_{k,i}^a$ we can suppose $n' = n$, $m_1' = m_1$ and $m_2' = m_2$, because otherwise they are empty. Therefore, we obtain

$$\frac{1}{|\Omega|} \int_{\Omega} |Df_{j+1} - Df_j|dx$$
$$\lesssim \sum_{i=1}^{j} \left(j^{-2} i^{-m_1'-2+\varepsilon'} + i^{-2+\varepsilon'} j^{m_2'-n'} + \sum_{a=m_2+1}^{n-m_1-1} \left[i^{a-n'}(j+1-i)^{-2} + j^{a-n'}(j+1-i)^{-2} \right] \right)$$
$$\lesssim j^{-2} + \sum_{i=1}^{j} i^{-2}(j+1-i)^{-2} \lesssim j^{-2}.$$

So iv) is proved. On the other hand, using $f_{j+1}(\omega_k) = g_k(\omega_k) = f_j(\omega_k)$ we have

$$\begin{aligned}
&\int_\Omega |Df_{j+1}^{-1} - Df_j^{-1}|dy = \sum_{k\in\mathbb{N}} \int_{g_k(\omega_k)} |Dg_k^{-1} - A_k^{-1}|dy \\
&\lesssim \sum_{\substack{k\in\mathbb{N}\\ A_k\in\bigcup_{i=1}^j A_j^i}} |g_k(\omega_k)| + \sum_{\substack{k\in\mathbb{N}\\ A_k\in\bigcup_{i=1}^j B_i^j}} j^{-2}|g_k(\omega_k)| + \sum_{\substack{k\in\mathbb{N}\\ A_k\in\bigcup_{i=1}^j\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a\cup S_{i,j}^a}} |g_k(\omega_k)| \\
&= \left|f_j\left(\Omega_{\bigcup_{i=1}^j A_j^i}\right)\right| + j^{-2}\left|f_j\left(\Omega_{\bigcup_{i=1}^j B_i^j}\right)\right| + \left|f_j\left(\Omega_{\bigcup_{i=1}^j\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,i}^a\cup S_{i,j}^a}\right)\right|.
\end{aligned}$$

Using again vi)–ix) and that we have a control over the determinant of $Df_j(x)$ when x is in the different sets, see (2.3.25), we obtain

$$\begin{aligned}
&\frac{1}{|\Omega|}\int_\Omega |Df_{j+1}^{-1} - Df_j^{-1}|dy \lesssim \sum_{i=1}^{j}\Big(j^{m_1'-n'}i^{m_1'}i^{-m_1'-2+\varepsilon'} + j^{-2}j^{n'-m_2'}i^{-m_2'}i^{-2+\varepsilon'}j^{m_2'-n'} \\
&\quad + \sum_{a=m_2+1}^{n-m_1-1}\Big[j^{-a}i^{n'-a}i^{a-n'}(j+1-i)^{-2} + i^{-a}j^{n'-a}j^{a-n'}(j+1-i)^{-2}\Big]\Big) \\
&\lesssim \left(j^{-2} + \sum_{i=1}^{j} i^{-2}(j+1-i)^{-2}\right) \lesssim j^{-2}.
\end{aligned}$$

Hence, we have proved v). Finally, we suppose that vi)–ix) hold for j and we prove them for $j+1$. Let C be a constant bigger than the ones appearing in Lemmas 2.3.9, 2.3.10 2.3.11 and 2.3.12.

Let $i = 1,\dots,j$; from the construction of ν_k we can see that if $\operatorname{supp}(\nu_k)\cap A_{j+1}^i \neq \emptyset$ then

$$\overline{\nu}_k = A_k \in \bigcup_{l=1}^{i} A_j^l \cup \bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,l}^a.$$

On the other hand, using part *(c)* of Proposition 1.3.5 we get

$$\begin{aligned}
|\Omega_{A_{j+1}^i}^{j+1}| &= \sum_{k:A_k\in\bigcup_{l=1}^i A_j^l\cup\bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,l}^a}\ \sum_{\ell:M_{k,\ell}\in A_{j+1}^i} \lambda_{k,\ell}|\omega_k| \\
&= \sum_{k:A_k\in\bigcup_{l=1}^i A_j^l\cup\bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S_{j,l}^a} \nu_k(A_{j+1}^i)|\omega_k| \\
&= \sum_{l=1}^{i}\sum_{k:A_k\in A_j^l} \nu_k(A_{j+1}^i)|\omega_k| + \sum_{l=1}^{i-1}\sum_{a=m_2+1}^{n-m_1-1}\sum_{k:A_k\in S_{j,l}^a} \nu_k(A_{j+1}^i)|\omega_k|.
\end{aligned}$$

Now, we use the control that we have over $\nu_k(A_{j+1}^i)$ given by Lemmas 2.3.9, 2.3.11 and 2.3.12, and also that given $S \subset \Gamma_+$ we have $\sum_{k:A_k\in S}|\omega_k| = |\Omega^j_S|$. Therefore, we obtain

$$\begin{aligned}|\Omega^{j+1}_{A^i_{j+1}}| &\leqslant \sum_{l=1}^{i-1}\sum_{k:A_k\in A^l_j} Cj^{2(m_1'+m_2'-n')}\frac{l^{m_1'}}{i^{m_1'+2}}|\omega_k| + \sum_{k:A_k\in A^i_j}|\omega_k| + \sum_{a=m_2+1}^{n-m_1-1}\sum_{k:A_k\in S^a_{j,i-1}} Ci^{a+m_1'-n'}|\omega_k| \\ &+ \sum_{l=1}^{i-2}\sum_{a=m_2+1}^{n-m_1-1}\sum_{k:A_k\in S^a_{j,l}} Cj^{2(m_2'-a)}\frac{l^{n'-a}}{i^{m_1'+2}}|\omega_k| = \sum_{l=1}^{i-1} Cj^{2(m_1'+m_2'-n')}\frac{l^{m_1'}}{i^{m_1'+2}}|\Omega^j_{A^l_j}| + |\Omega^j_{A^i_j}| \\ &+ \sum_{a=m_2+1}^{n-m_1-1} Ci^{a+m_1'-n'}|\Omega^j_{S^a_{j,i-1}}| + \sum_{l=1}^{i-2}\sum_{a=m_2+1}^{n-m_1-1} Cj^{2(m_2'-a)}\frac{l^{n'-a}}{i^{m_1'+2}}|\Omega^j_{S^a_{j,l}}|.\end{aligned}$$

Let $M_j = \max_{i=1,\dots,j} C^1_{j,i}$, as in Lemma 2.3.13. By induction we get

$$\begin{aligned}\frac{|\Omega^{j+1}_{A^i_{j+1}}|}{|\Omega|} &\leqslant \sum_{l=1}^{i-1} C^1_{j,l}Cj^{2(m_1'+m_2'-n')}\frac{l^{m_1'}}{i^{m_1'+2}}l^{-m_1'-2+\varepsilon'} + C^1_{j,i}i^{-m_1'-2+\varepsilon'} \\ &+ \sum_{a=m_2+1}^{n-m_1-1} C^2_{j,i-1}Ci^{a+m_1'-n'}(i+2)^{a-n'}(j+2-i)^{-2} \\ &+ \sum_{l=1}^{i-2}\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,l}Cj^{2(m_2'-a)}\frac{l^{n'-a}}{i^{m_1'+2}}(l+2)^{a-n'}(j+1-l)^{-2}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{|\Omega^{j+1}_{A^i_{j+1}}|}{|\Omega|} &\leqslant \Big(C^1_{j,i} + j^{-2}2CM_j + nCC^2_{j,i-1}(j+2-i)^{-2} \\ &+ \sum_{l=1}^{i-2} 2CC^2_{j,l}j^{-2}(j+1-l)^{-2}\Big) i^{-m_1'-2+\varepsilon'} \leqslant C^1_{j+1,i}i^{-m_1'-2+\varepsilon'}.\end{aligned}$$

The last estimate comes from *c)* of Lemma 2.3.13; we will use this lemma several times in the rest of the proof with $\tilde{C} = 16^nC$. Now, for each $i = 1,\dots,j$, and $k\in\mathbb{N}$ such that $\operatorname{supp}(\nu_k)\cap B^{j+1}_i \neq \emptyset$, we have

$$\overline{\nu}_k = A_k \in \bigcup_{l=1}^{i} B^j_l \cup \bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S^a_{l,j},$$

hence, proceeding as before we get

$$\begin{aligned}\frac{|\Omega^{j+1}_{B^{j+1}_i}|}{|\Omega|} &\leqslant \sum_{l=1}^{i-1} C^1_{j,l}C(i(j+3))^{2(m_1'+m_2'-n')}l^{-2+\varepsilon'}(j+2)^{m_2'-n'} \\ &+ C^1_{j,i}\left(\left(\frac{j+2}{j+3}\right)^{n'-m_2'} + C(ij)^{-2}\right)i^{-2+\varepsilon'}(j+2)^{m_2'-n'} \\ &+ \sum_{a=m_2+1}^{n-m_1-1} C^2_{j,i-1}C(i^2j)^{m_2'-a}(j+2)^{a-n'}(j+2-i)^{-2} \\ &+ \sum_{l=1}^{i-2}\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,l}Ci^{2(m_1'+m_2'-n')}j^{2m_1'+m_2'-2n'+a}(j+2)^{a-n'}(j+2-l)^{-2}.\end{aligned}$$

Therefore

$$\frac{|\Omega^{j+1}_{B_i^{j+1}}|}{|\Omega|} \leqslant \Big(C^1_{j,i} + j^{-2}2^{n+1}CM_j + n2^{2n}CC^2_{j,i-1}(j+2-i)^{-2} + \sum_{l=1}^{i-1} C^2_{j,l}2^nCj^{-2}(j+2-l)^{-2}\Big) i^{-2+\varepsilon'}(j+3)^{m_2'-n'} \leqslant C^1_{j+1,i}\, i^{-2+\varepsilon'}(j+3)^{m_2'-n'}.$$

Now, we use that those $k\in\mathbb{N}$ such that ν_k satisfy

$$\operatorname{supp}(\nu_k)\cap A^{j+1}_{j+1}\neq\emptyset \text{ or } \operatorname{supp}(\nu_k)\cap B^{j+1}_{j+1}\neq\emptyset$$

are those that satisfy

$$\overline{\nu}_k = A_k \in \bigcup_{l=1}^{j}\left(A^l_j\cup B^j_l\cup \bigcup_{a=m_2+1}^{n-m_1-1}\left(S^a_{l,j}\cup S^a_{j,l}\right)\right).$$

Therefore, we obtain

$$\begin{aligned}\frac{|\Omega^{j+1}_{A_{j+1}^{j+1}}|}{|\Omega|} \leqslant{}& \sum_{l=1}^{j} C^1_{j,l}C\left(j^{2(m_1'+m_2'-n')}\frac{l^{m_1'}}{j^{m_1'+2}}l^{-m_1'-2+\varepsilon'} + j^{m_1'+m_2'-n'}l^{-2+\varepsilon'}(j+2)^{m_2'-n'}\right)\\ &+\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,j}Cj^{a+m_1'-n'}(j+2)^{a-n'}\\ &+\sum_{l=1}^{j-1}\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,l}C\left(j^{2(m_2-a)}\frac{l^{n'-a}}{j^{m_1'+2}}(l+2)^{a-n'}(j+1-l)^{-2} + j^{a+m_1'-n'}(j+2)^{a-n'}(j+1-l)^{-2}\right),\end{aligned}$$

so

$$\begin{aligned}\frac{|\Omega^{j+1}_{A_{j+1}^{j+1}}|}{|\Omega|} \leqslant{}& M_j4Cj^{-m_1'-2} + nCC^2_{j,j}j^{-m_1'-2}\\ &+\sum_{l=1}^{j-1} CC^2_{j,l}\left(j^{-2}(j+1-l)^{-2}+(j+1-l)^{-2}\right)j^{-m_1'-2}\\ ={}& C\left(4M_j + nC^2_{j,j} + \sum_{l=1}^{j-1}C^2_{j,l}\left(j^{-2}(j+1-l)^{-2}+(j+1-l)^{-2}\right)\right)j^{-m_1'-2}\\ \leqslant{}& C^1_{j+1,j+1}(j+1)^{-m_1'-2+\varepsilon'}.\end{aligned}$$

We also get

$$\begin{aligned}\frac{|\Omega^{j+1}_{B_{j+1}^{j+1}}|}{|\Omega|} \leqslant{}& \sum_{l=1}^{j} C^1_{j,l}C\left(j^{2m_1'+3m_2'-3n'}l^{m_1'}l^{-m_1'-2+\varepsilon'} + (j(j+1))^{2(m_1'+m_2'-n')}l^{-2+\varepsilon}(j+2)^{m_2'-n'}\right)\\ &+\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,j}Cj^{3(m_2'-a)}(j+2)^{a-n'}\\ &+\sum_{l=1}^{j-1}\sum_{a=m_2+1}^{n-m_1-1} CC^2_{j,l}\left(j^{3m_2'-n'-2a}l^{n'-a}(l+2)^{a-n'}(j+1-l)^{-2}\right.\\ &\left.+j^{2(m_1'+m_2'-n')}j^{2m_1'+m_2'-2n'+a}(j+2)^{a-n'}(j+1-l)^{-2}\right).\end{aligned}$$

Therefore

$$\frac{|\Omega^{j+1}_{B^{j+1}_{j+1}}|}{|\Omega|} \leqslant M_j 4Cj^{m_2'-n'-2} + nCC^2_{j,j} j^{m_2'-n'-2} \quad + Cj^{m_2'-n'-2} \sum_{l=1}^{j-1} C^2_{j,l}\left((j+1-l)^{-2} + j^{-2}(j+1-l)^{-2}\right)$$
$$= \left(4CM_j + 2nC\sum_{l=1}^{j} C^2_{j,l}(j+1-l)^{-2}\right) j^{m_2'-n'-2} \leqslant C^1_{j+1,j+1}(j+1)^{-2+\varepsilon'}(j+3)^{m_2'-n'}.$$

Hence, vi) and vii) are proved for $j+1$.

Finally, proceeding as before, we only have to prove viii) and ix) for $j+1$. For each $i=1,\dots,j$, $a=m_2+1,\dots,n-m_1-1$ and $k\in\mathbb{N}$ such that $\operatorname{supp}(v_k)\cap S^a_{j+1,i}\neq\emptyset$, we have

$$\overline{v}_k = A_k \in \bigcup_{l=1}^{i} A^l_j \cup \bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S^a_{j,l},$$

and

$$\frac{|\Omega^{j+1}_{S^a_{j+1,i}}|}{|\Omega|} \leqslant \sum_{l=1}^{i-1} C^1_{j,l} C j^{2(m_1'+m_2'-n')} l^{m_1'} i^{2m_2'-a-n'} l^{-m_1'-2+\varepsilon'} + C^1_{j,i} C(j^2 i)^{m_1'+a-n'} i^{-m_1'-2+\varepsilon'}$$
$$+ \sum_{b=m_2+1}^{a-1} C^2_{j,i-1} C(i-1)^{b-a}(i+1)^{b-n'}(j+2-i)^{-2}$$
$$+ C^2_{j,i-1}\left(\left(\frac{i+1}{i+2}\right)^{n'-a} + C(j(i-1))^{-2}\right)(i+1)^{a-n'}(j+2-i)^{-2}$$
$$+ \sum_{b=a+1}^{n-m_1-1} C^2_{j,i-1} C(j^2(i-1))^{a-b}(i+1)^{b-n'}(j+2-i)^{-2}$$
$$+ \sum_{l=1}^{i-2}\sum_{b=m_2+1}^{n-m_1-1} C^2_{j,l} C j^{2(m_2'-b)} l^{n'-b}(j+1)^{2m_2'-a-n'}(l+2)^{b-n'}(j+1-l)^{-2},$$

hence, using $2m_2-a-n\leqslant a-n-2$ for $a\geqslant m_2+1$ we get

$$\frac{|\Omega^{j+1}_{S^a_{j+1,i}}|}{|\Omega|} \leqslant \left(C^2_{j,i-1} + n4^n CC^2_{j,i-1} j^{-2} + 4^n CM_j i^{-2+\varepsilon'} + nCj^{-4}\sum_{l=1}^{i-2} C^2_{j,l}\right)(i+2)^{a-n'}(j+2-i)^{-2}$$
$$\leqslant C^2_{j+1,i}(i+2)^{a-n'}(j+2-i)^{-2}.$$

If $\operatorname{supp}(v_k)\cap S^a_{i,j+1}\neq\emptyset$, then

$$\overline{v}_k = A_k \in \bigcup_{l=1}^{i} B^j_l \cup \bigcup_{l=1}^{i-1}\bigcup_{a=m_2+1}^{n-m_1-1} S^a_{l,j},$$

and

$$\begin{aligned}\frac{|\Omega^{j+1}_{S^a_{i,j+1}}|}{|\Omega|} \leqslant &\sum_{l=1}^{i-1} C^1_{j,l} C j^{m_1'+m_2'-n'} \left((i-1)^2(j+1)\right)^{m_1'+a-n'} l^{-2+\varepsilon'}(j+2)^{m_2'-n'}\\ &+ C^1_{j,i} C j^{m_2'-a} i^{-2+\varepsilon'}(j+2)^{m_2'-n'} + \sum_{b=m_2+1}^{a-1} C^2_{j,i-1} C j^{b-a}(j+2)^{b-n'}(j+2-i)^{-2}\\ &+ C^2_{j,i-1}\left(\left(\frac{j+2}{j+3}\right)^{n'-a} + C((i-1)j)^{-2}\right)(j+2)^{a-n'}(j+2-i)^{-2}\\ &+ \sum_{b=a+1}^{n-m_1-1} C^2_{j,i-1} C((i-1)^2 j)^{a-b}(j+2)^{b-n'}(j+2-i)^{-2}\\ &+ \sum_{l=1}^{i-2}\sum_{b=m_2+1}^{n-m_1-1} C^2_{j,l} C j^{2m_1'-2n'+a+b}(i-1)^{2(m_1'+a-n)}(j+2)^{b-n'}(j+2-l)^{-2},\end{aligned}$$

so, using again $2m_2-a-n \leqslant a-n-2$ for $a \geqslant m_2+1$ and that $a \leqslant n-m_1-1$ we obtain

$$\begin{aligned}\frac{|\Omega^{j+1}_{S^a_{i,j+1}}|}{|\Omega|} &\leqslant \left(C^2_{j,i-1} + n8^n C C^2_{j,i-1} i^{-2} + 8^n C M_j i^{-2+\varepsilon'} + n8^n C i^{-2} j^{-2}\sum_{l=1}^{i-2} C^2_{j,l}\right)(j+3)^{a-n'}(j+2-i)^{-2}\\ &\leqslant C^2_{j+1,i}(j+3)^{a-n'}(j+2-i)^{-2}.\end{aligned}$$

It only remains to estimate $|\Omega^{j+1}_{S^a_{j+1,j+1}}|$ for $a \in \{m_2+1,\dots,n-m_1-1\}$. If $\operatorname{supp}(v_k) \cap S^a_{j+1,j+1} \neq \emptyset$, then

$$\overline{v}_k = A_k \in \bigcup_{l=1}^{j}\left(A^l_j \cup B^j_l \cup \bigcup_{a=m_2+1}^{n-m_1-1}\left(S^a_{j,l} \cup S^a_{l,j}\right)\right).$$

Therefore, using $2m_2-a-n \leqslant a-n-2$ for $a \geqslant m_2+1$ we have

$$\begin{aligned}\frac{|\Omega^{j+1}_{S^a_{j+1,j+1}}|}{|\Omega|} \leqslant &\sum_{l=1}^{j} C^1_{j,l} C\Big(j^{2(m_1'+m_2'-n')} l^{m_1'}(j+1)^{2m_2'-a-n'} l^{-m_1'-2+\varepsilon'}\\ &+ j^{m_1'+m_2'-n'}\left(j^2(j+1)\right)^{m_1'+a-n'} l^{-2+\varepsilon'}(j+2)^{m_2'-n'}\Big) + \sum_{b=m_2+1}^{a-1} C^2_{j,j} C j^{b-a}(j+2)^{b-n'}\\ &+ C^2_{j,j}\left(\left(\frac{j+2}{j+3}\right)^{n'-a} + C(j^2)^{-2}\right)(j+2)^{a-n'} + \sum_{b=a+1}^{n-m_1-1} C^2_{j,j} C(j^3)^{a-b}(j+2)^{b-n'}\\ &+ \sum_{l=1}^{j-1}\sum_{a=m_2+1}^{n-m_1-1} C^2_{j,l} C\left(j^{2(m_2'-b)} l^{n'-b} j^{2m_2'-a-n'} + j^{2m_1'-2n'+a+b} j^{2(m_1'+a-n')}\right)(j+2)^{b-n'}\\ \leqslant &\left(C^2_{j,j} + n8^n C C^2_{j,j} j^{-2} + 8^n C M_j j^{-4} + n8^n C j^{-4}\sum_{l=1}^{j-1} C^2_{j,l}\right)(j+3)^{a-n'} \leqslant C^2_{j+1,j+1}(j+3)^{a-n'}.\end{aligned}$$

Then we get viii)-ix), and, hence, we have proved i)-ix). Since $Df \in \Gamma_+$ a.e. in Ω we can use Lemma 1.3.1, hence, there exists a convex function $u \in W^{2,1}$ such that $f = \nabla u$.

Following the same reasoning as in 2.2.1, one can show that our function u is strictly convex.

□

Chapter 3

Invertibility and relaxation in nonlinear elasticity

In Chapter 2 we have dealt with Sobolev homeomorphisms with pathological properties. At the root of those properties are the lack of Luzin's conditions N and N^{-1} and the fact that the distributional determinant Det differs from the pointwise determinant det. In the other direction, the study of Sobolev homeomorphisms (or, in general, Sobolev maps) that do not present such pathologies leads naturally to the study of regularity properties of Sobolev maps, which is a subject that has received an immense attention since the pioneering works of Sobolev, Morrey, Gagliardo and Nirenberg, and is still an intense area of research. In the following paragraphs we mention some landmarks in this direction that are relevant to our work.

We start with the pioneering result of Morrey [118], stating that a Sobolev map in $W^{1,p}$ with $p > n$ has a representative that is Hölder continuous. Marcus and Mizel [113] proved that the continuous representative satisfies Luzin's condition N. It had been proved before that they are differentiable a.e.: Cesari [26] proved it for $n = 2$ and Calderón [22] extended the result to a general n.

As for the borderline case $p = n$, Vodopýanov and Goldshtein [153] showed that a Sobolev map $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ with $\det Du > 0$ a.e. (or, in general, of finite distortion, i.e., $Du(x) = 0$ for a.e. x for which $\det Du(x) = 0$) has a continuous representative that satisfies Luzin's condition N and whose components are weakly monotone. It was also proved later [140] that they are differentiable a.e.

The first regularity results for the case $p < n$ were done by Šverák [156] motivated by the existence results in Nonlinear Elasticity by Ball [10]. Šverák proved that if $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $\operatorname{cof} Du \in L^q(\Omega, \mathbb{R}^{n\times n})$ and $\det Du > 0$ a.e. with $n-1 < p \leqslant n$ and $q \geqslant \frac{p}{p-1}$, then u has a representative that is continuous except in a set of p-capacity zero (although later [73] it was shown that in fact they are continuous $\mathcal{H}^{n-p}$ a.e.) and satisfies Luzin's condition N. The class of functions $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $\operatorname{cof} Du \in L^q(\Omega, \mathbb{R}^{n\times n})$ and $\det Du > 0$ a.e. will be denoted in this introduction as $\mathscr{A}_{p,q}(\Omega)$. The same conclusion was achieved by Müller, Qi and Yan [122] assuming only $q \geqslant \frac{n}{n-1}$.

The issue of continuity $\mathcal{H}^{n-p}$ a.e. and differentiability a.e. was clarified and unified with the concept of *monotonicity*, which is an old notion going back to Lebesgue [104] (see also

[119]). This notion was adapted by Manfredi [112] to Sobolev functions: he showed that monotone $W^{1,p}$ functions are continuous except in a set of p-capacity zero (again, it was proved later [73] that they are continuous $\mathscr{H}^{n-p}$ a.e.). Manfredi's work gave rise to many generalizations (see, e.g., the monograph [86]). We mention, in particular, Hajłasz and Malý [73], where they proved that a slightly weaker version of monotonicity implies the continuity $\mathscr{H}^{n-p}$ a.e. and the differentiability a.e. (following an earlier work by [151]). They also showed that maps $u \in \mathscr{A}_{p,q}(\Omega)$ with $n-1 < p \leqslant n$ and $q \geqslant \frac{n}{n-1}$ are monotone, so proving at once in a unified way the regularity properties in $\mathscr{A}_{p,q}$.

On the other hand, it has been an active field of research to find conditions on a Sobolev map guaranteeing that

$$\operatorname{Det} Du = \det Du. \tag{3.0.1}$$

This equality has been traditionally an intermediate step to prove existence theorems in nonlinear Elasticity [10], hence the motivation to its study. Moreover, equality (3.0.1) means that u does not present the phenomenon of cavitation (the formation of voids, see [123, 143, 31, 78, 16]), hence a higher regularity of u is expected. When the discontinuities of u produced by cavitation are excluded.

Resetnjak [137, 138] proved (3.0.1) for $W^{1,n}$ functions, as well as the continuity of Det and det. Resetnjak's results were largely ignored in the Western mathematical community. In particular, Ball [10] proves, independently of Resetnjak, that $\operatorname{Det} Du = \det Du$ for functions $u \in W^{1,p}$ with $p \geqslant n$. He also proved it for functions $u \in \mathscr{A}_{p,q}$ with $p \geqslant n-1$ and $q \geqslant \frac{p}{p-1}$. Müller, Qi and Yan [122] were able to extend this result to $q \geqslant \frac{n}{n-1}$. In [120] Müller proved that (3.0.1) holds for mappings $u \in W^{1,\frac{n^2}{n+1}}$ such that $\operatorname{Det} Du \in L^1$. In [95] the authors showed that for (3.0.1) to hold it is enough that $|Du|^n \in L\log L$ and $\det Du \geqslant 0$. This result was extended by Greco in [68] to a slightly bigger space than $L\log L$. In [42], De Lellis proved that for functions in the class $B_n V$, the distributional determinant can be decomposed in a lower dimensional part, an absolutely continuous part and a Cantor part. Later on, it was proved by De Lellis himself and Ghiraldin [43] that the absolutely continuous part is the pointwise determinant. In [78] it was proved that (3.0.1) holds under the assumptions that $u \in W^{1,n-1} \cap L^\infty$ satisfies the condition INV and $\det Du > 0$ a.e. Recently, it was proved [49] that in order u to satisfy (3.0.1) it is enough that $u \in W^{1,n-1}$ satisfies Luzin's condition and $\det Du \in L^1$. In [16] the authors prove that the functions in the class $\mathscr{A}_p$, which we will discuss later, satisfy (3.0.1).

As a conclusion of the previous paragraphs, we can see that there is a remarkable parallelism between the following properties that a Sobolev $W^{1,p}$ map u can possess: satisfaction of Luzin's condition N, equality $\operatorname{Det} Du = \det Du$, monotonicity, continuity $\mathscr{H}^{n-p}$ a.e., and differentiability a.e.

Another issue related to the regularity of Sobolev maps with $\det Du > 0$ a.e. is the satisfaction of some invertibility property. This question amounts to asking for an inverse function theorem analogue for Sobolev maps. Rather than local invertibility, most of the work in this direction within the community of Nonlinear elasticity has been focused on global invertibility. The motivation is that global invertibility prevents interpenetration of matter, which is a property that any elastic deformation must have. An additional issue is a proper definition of *invertibility* in this context, since being u a homeomorphism is an unrealistic assumption (and conclusion) in nonlinear elasticity, because of the common presence of singularities (which prevents the continuity) and self-contact (which prevents the injectivity everywhere).

In his pioneering paper, Ball [11] proved sufficient conditions guaranteeing that the deformation is a homeomorphism, and, moreover, established weaker assumptions for the invertibility condition

$$\operatorname{Card} u^{-1}(y) = 1 \quad \text{a.e. } y \in u(\Omega), \tag{3.0.2}$$

notably, that $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n$, $\det Du > 0$ a.e. and u coincides on $\partial\Omega$ with an injective map.

Later, Šverák [156] showed that the invertibility condition (3.0.2) holds with the weaker integrability $p > n-1$ and $q \geqslant \frac{p}{p-1}$. He also deduced another notion of invertibility in this class, namely, the set of points $x \in \Omega$ such that the topological image by u^{-1} of the topological image by u of x has more than two points. This apparently involved definition appeared naturally in the development of his theory.

An alternative approach was developed by Ciarlet and Nečas [30]. They imposed the condition

$$\int_\Omega \det Du \, dx \leqslant |u(\Omega)|$$

and proved, when $p > n$, that this implies condition (3.0.2). Tang [134] generalized the results of [30] to the case $p > n-1$ and $q \geqslant \frac{p}{p-1}$ by using the techniques of [156]. Both [156] and [30] were generalized by [122] for the case $p > n-1$ and $q \geqslant \frac{n}{n-1}$. A further generalization of [30], beyond Sobolev spaces, was done by Giacomini and Ponsiglione [65] in the space SBV of special functions of bounded variation, so as to allow for fracture in the materials.

In their study of cavitation, Müller and Spector [123] worked with the natural concept of *injective a.e.*, which means that u is injective in a subset of Ω of full measure. Of course, condition (3.0.2) is equivalent to injectivity a.e., provided Luzin's conditions N and N^{-1} hold, as is the case for the maps object of the study. Actually, the novel invertibility concept introduced in [123] is condition INV. Condition INV is a topological property that involves the topological degree, which is stronger than injective a.e. and ensures that, for a.e. ball $B \subset \Omega$, material inside ∂B goes to material inside $u(\partial B)$, and material outside ∂B goes to material outside $u(\partial B)$. Roughly, *almost every sphere is impenetrable.* His motivation for this new condition was that, as they observed (see [123, 124], and, later, [78] for some pathological examples), conditions injectivity a.e. and $\det Du > 0$ a.e. do not prevent interpenetration of matter or orientation reversal. Condition INV has sequentially been adopted by many authors studying cavitation as the right condition of invertibility in Nonlinear elasticity [143, 31, 144, 75, 78, 116, 90, 19]. Injectivity a.e. in SBV (a space where condition INV cannot be defined because of the absence of a degree) was studied in [76, 77, 80].

In contrast to the previous paragraphs, studies on local invertibility in the context of Nonlinear elasticity are fewer. We are only aware of those of [62] and [16]. Fonseca and Gangbo [62] proved a version of the inverse function theorem for Sobolev maps $W^{1,p}(\Omega, \mathbb{R}^n)$ with $\det Du > 0$ a.e. for $p \geqslant n$. This was generalized in Barchiesi, Henao and Mora-Corral [16] for $p > n-1$. In the context of geometric function theory, studies on local invertibiliy started with the pioneering results of Rešetnjak [136] and, much later, Iwaniec and Šverák [96]. In this remarkable paper, [96], they prove that, when $n=2$, if $u \in W^{1,n}(\Omega,\mathbb{R}^n)$ satisfies $\det Du \geqslant 0$ a.e. and has dilatation in L^{n-1} then u is open and discrete (which is, in turn, a local invertibility property). The validity of the previous result for $n \geqslant 3$ is known as the Iwaniec-Šverák conjecture. Paper [96] has undergone many generalizations [74, 98, 84, 92, 150], but the Iwaniec-Šverák conjecture remains unsolved.

Another related issue is to ascertain the regularity of the inverse, once its existence has been proved. Since this is an aside question in this thesis, here we only mention the results of [38, 81, 85, 87, 88, 130, 152] in the context of Geometric function theory and [11, 156, 77, 79, 16] in the context of Nonlinear elasticity. In this chapter we will only use the fact that in our class of maps $\mathscr{A}_p$, the inverse u^{-1}, properly defined, is Sobolev $W^{1,1}$.

The starting point of Chapter 3 is the class $\mathscr{A}_p$ introduced by [16] as the set of $u \in W^{1,p}(\Omega,\mathbb{R}^n)$, $p > n-1$, such that $\det Du \in L^1_{\mathrm{loc}}$, $\det Du > 0$ a.e. and $\mathscr{E}(u)=0$. Here $\mathscr{E}(u)$ is the area of the new surface created by u (see [76, 77] or Definition 3.2.13 below). The equality $\mathscr{E}(u)=0$ says that u does not create new surface; for the purposes of this chapter, it is enough to know that $\mathscr{E}(u)=0$ holds if and only if

$$\operatorname{div} g(u(x)) \det Du(x) = \operatorname{Div}[\operatorname{adj} Du(x)\, g(u(x))] \tag{3.0.3}$$

for all $g \in C^1(\mathbb{R}^n,\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$. The operator Div in the right-hand side denotes the distributional divergence in the reference configuration. Observe that equality (3.0.3) implies $\operatorname{Det} = \det$, just by taking $g = \frac{1}{n}\operatorname{id}$. In fact, as shown in [16], condition (3.0.3) is only slightly stronger that $\operatorname{Det} = \det$. In particular, maps in $\mathscr{A}_p$ do not create cavities. They proved existence of minimizers in the class $\mathscr{A}_p$, a local inverse function theorem, as well as regularity properties of the deformations in $\mathscr{A}_p$, notably, Luzin's condition N and monotonicity, which implies continuity $\mathscr{H}^{n-p}$ a.e. and differentiability a.e. To have an idea of how big $\mathscr{A}_p$ is, we mention that, as a consequence of [122], $\mathscr{A}_{p,q} \subset \mathscr{A}_p$ for $p > n-1$ and $q \geqslant \frac{n}{n-1}$.

In this chapter we first generalize and unify the above-mentioned global invertibility results. We show that, in the class $\mathscr{A}_p$ (which is larger than those studied in [11], [30], [156], [134] and [122]) all approaches of invertibility are equivalent. Our proof is a follow-up of that of [16] on local invertibility.

The second part of this chapter is an instance of a general property that we think $\mathscr{A}_p$ has: most properties that are true in $W^{1,p}$ for $p > n$ also hold in $\mathscr{A}_p$ with $p > n-1$. To be precise, we show a relaxation result in nonlinear elasticity set in $\mathscr{A}_p$. The word *relaxation* in the context of Calculus of Variations refers to the lower semicontinuous envelope, i.e., the largest lower semicontinuous functional (in the appropriate topology) below a given one. It is a classical result going back to Young [163] that, in the weak topology of L^p, the relaxation of

$$\int_\Omega W(u)\,dx \quad \text{is} \quad \int_\Omega W^c(u)\,dx$$

where W^c is the *convexification* of W, i.e., the largest convex function below W. Modern expositions of this fact can be found, e.g., in [51, 21, 63, 40].

It is also well-known [39] that the relaxation of a functional of the type $\int_\Omega W(Du)\,dx$, in the weak topology of $W^{1,p}$, is $\int_\Omega W^{qc}(Du)\,dx$, where W^{qc}, the *quasiconvexification* of W, is the largest quasiconvex function below W. However, neither this result nor its many generalizations (see, e.g., [17, 70, 71, 40, 160, 145, 146, 111]) meet the growth assumptions in nonlinear elasticity, in which the stored energy function $W:\mathbb{R}^{n\times n}\to\mathbb{R}\cup\{\infty\}$ is required to satisfy

$$W(F)=\infty \text{ if } \det F\leqslant 0 \quad\text{and}\quad W(F)\to\infty \text{ as } \det F\to 0, \tag{3.0.4}$$

so as to avoid orientation reversal.

Recently, Conti and Dolzmann [33] established the first result of relaxation compatible with the growth condition (3.0.4). They proved it for $W^{1,p}$ deformations with $p\geqslant n$. They also suppose, as the main assumption, that W^{qc} is polyconvex, so as to obtain the lower semicontinuity. Indeed, although the necessary and sufficient condition for lower semicontinuity under p-growth conditions is the quasiconvexity of the integrand [117, 1], there are no proofs so far that meet the growth condition (3.0.4); nevertheless, polyconvexity is compatible with (3.0.4) and provides a general sufficient condition for lower semicontinuity [12]. In this chapter we generalize the relaxation result of [33] to cover the class $\mathscr{A}_p$ and hence, lower the exponent from $p\geqslant n$ to $p>n-1$.

We also cover energies of the form

$$\int_\Omega W(Du,\vec{n}(u))\,dx+\int_{u(\Omega)}|D\vec{n}(y)|^2\,dy. \tag{3.0.5}$$

This type of energies appears in [15], [47] and [161] to model liquid crystal nematic elastomers. Nematic elastomers are a type of liquid crystals elastomers, which are a kind of material that combines the properties of liquid crystals and rubber-like solids. Their inner structure is made by a network of cross-linked polymer chains. In those chains, elongated rigid monomer units are incorporated or attached sideways. If the order of those chains is uniaxial and the degree of the order is fixed, their orientational order is described by a director field $\vec{n}$ of norm 1 defined in the deformed configuration; it describes the direction of alignment of the molecules at $u(x)$. This vector field is the key to understand the anisotropic behaviour of the material. The first term of the energy (3.0.5) is the mechanical energy, which couples the elastic energy of the deformation with the director field. The second term penalizes the spatial non-uniformity of directors. Both make up the energy of the pair deformation-orientation $(u,\vec{n})$. In [16] it was proved the existence of minimizers of (3.0.5) under the assumption that W is polyconvex in its first variable. In this chapter we show that if W is not even quasiconvex, the relaxation of (3.0.5) in the class $\mathscr{A}_p$, with the weak topology of $W^{1,p}$, is

$$\int_\Omega W^{qc}(Du,\vec{n}(u))\,dx+\int_{u(\Omega)}|D\vec{n}(y)|^2\,dy, \tag{3.0.6}$$

where W^{qc} is the quasiconvexification of W with respect to the first variable. The main assumption is, as in [33], that W^{qc} is polyconvex.

The structure of the chapter is the following. In Section 3.1 we give the general notation that we will use in this chapter. In Section 3.2 we define the key concepts of this chapter; in particular, we define the condition INV and the class $\mathscr{A}_p$. In Section 3.3 we prove some auxiliary results for the functions in the class $\mathscr{A}_p$; in particular, we prove a representation of $\mathscr{E}(u,f)$ as a surface integral (see Definition 3.2.13), and that given $B\subset\subset\Omega$, $u\in\mathscr{A}_p(B)$ and $v\in\mathscr{A}_p(\Omega)$ that coincide in a neighbourhood of ∂B, the function defined as u in B and as v in $\Omega\setminus B$ is in $\mathscr{A}_p$. In Section 3.4 we prove that in the class $\mathscr{A}_p$ all the notions of invertibility that were explained above in this introduction are equivalent. Finally, in Section 3.5 we prove that, in the class $\mathscr{A}_p$, the relaxation of (3.0.5) is (3.0.6).

3.1 Notation of Chapter 3

We explain the general notation used throughout this chapter, most of which is standard.

- In the whole chapter, Ω is an open, non-empty bounded set of $\mathbb{R}^n$, which plays the role of the reference configurations of the elastic body. Sometimes Ω will be required to have Lipschitz regularity.
- We denote by $SL(n)$ the multiplicative subgroup of matrices $M \in \mathbb{R}^{n\times n}$ with $\det M = 1$ and by $\mathbb{R}^{n\times n}_+$ the matrices with positive determinant.
- We will use the symbol $\lesssim$ when there exists a constant depending only on n such that the left hand side is less than or equal to the constant times the right hand side.
- Given a set $E \subset \mathbb{R}^n$, we denote its characteristic function by χ_E. We write $\operatorname{Card} E$ for the number of elements of E. When it is measurable, its Lebesgue measure is denoted by $|E|$ and we use $\mathscr{H}^m(E)$ for its Hausdorff measure of dimension m.
- The identity function is denoted by id and the Sobolev space from Ω to $\mathbb{R}^n$ is denoted, alternatively, by $W^{1,p}$, $W^{1,p}(\Omega)$ or $W^{1,p}(\Omega,\mathbb{R}^n)$.
- $u : \Omega \to \mathbb{R}^n$ is the deformation of the body. It will be required to be in $W^{1,p}$ with some additional properties.
- We will use the notation $\vec{n}$ for a function in the deformed configuration that typically has $u(x)$ as its argument and takes values in $\mathbb{S}^{n-1}$, the set of vectors in $\mathbb{R}^n$ of norm 1. The norm of a $v \in \mathbb{R}^n$ is denoted by $|v|$.
- Given $A \in \mathbb{R}^{n\times n}$, its operator norm is denoted by $|A|$, which coincides with the highest singular value.
- We use $\cdot$ to denote the inner product (componentwise) of matrices.
- We denote by $\operatorname{adj} A$ and $\operatorname{cof} A$ the adjugate and cofactor matrices of $A \in \mathbb{R}^{n\times n}$, respectively, i.e., $(\det A)I = A \operatorname{adj} A$ and $\operatorname{cof} A = (\operatorname{adj} A)^T$. Observe that the cofactor satisfies $\operatorname{cof}(AB) = \operatorname{cof}(A)\operatorname{cof}(B)$ for $A, B \in \mathbb{R}^{n\times n}$.
- Given a function $u : \overline{\Omega} \to \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus u(\partial\Omega)$ we denote by $\deg(u,\Omega,y)$ the topological degree of the function u in Ω at y.
- We will use the symbol $\rightharpoonup$ to denote the weak convergence.
- We denote by $p' = \frac{p}{p-1}$ the Hölder conjugate exponent of $p \geqslant 1$, and by

$$p^* = \begin{cases} \frac{pn}{n-p} & \text{if } 1 \leqslant p < n, \\ \infty & \text{if } n \leqslant p \end{cases}$$

  the Sobolev conjugate exponent of $p \geqslant 1$.

3.2 Definitions

The aim of this section is to define the concepts needed in this chapter and to state some preliminary results. Some of them are well known in the theory of weakly differentiable functions [59, 53, 164] and some of them are specific of the theory of cavitation in nonlinear elasticity. In the whole chapter the exponent $p > n-1$ is fixed. The main concepts to be defined are the following.

- The class of functions $\mathscr{A}_p(\Omega)$, which consists of the functions $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that $\det Du > 0$, $\det Du \in L^1_{\mathrm{loc}}$ and for all $g \in C^1(\mathbb{R}^n, \mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, u satisfies

$$\operatorname{div} g(u(x)) \det Du(x) = \operatorname{Div}[\operatorname{adj} Du(x) g(u(x))],$$

  where Div is the distributional divergence. For smooth maps u, the last equality with $g = \frac{1}{n}\,\mathrm{id}$ is a consequence of Piola's identity. The variant presented above was introduced in [120, 122].

- The class of good open sets $\mathscr{U}_u$, which consists of enough open sets $U \subset\subset \Omega$ where, among other things, the topological degree of u is well defined and $u \in W^{1,p}(\partial U)$.

- The condition INV, which is one of the many approaches to obtain the existence of the inverse of a function. In Theorem 3.4.1 we will prove that all the approaches are equivalent in the class $\mathscr{A}_p$.

To define these concepts we need some auxiliary definitions. The reader only interested in these concepts may omit the rest of the section.

The *density* $D(A, x)$ of a measurable set $A \subset \mathbb{R}^n$ at an $x \in \mathbb{R}^n$ is defined as

$$D(A, x) := \lim_{r \to 0} \frac{|A \cap B(x, r)|}{|B(x, r)|}.$$

In this chapter we do not identify functions that coincide a.e.

The following definition is due to Federer [59] (see also [3, Definition 4.31] or [123, Definition 2.3]).

Definition 3.2.1. *Let $u : \Omega \to \mathbb{R}^n$ be a measurable function, and consider $x_0 \in \Omega$.*

a) We say that u is approximately differentiable at x_0 if u is defined at x_0 and there exists $F \in \mathbb{R}^{n\times n}$ such that

$$D\left(\left\{x \in \Omega \setminus \{x_0\} : \frac{|u(x) - u(x_0) - F(x - x_0)|}{|x - x_0|} \geqslant \delta\right\}, x_0\right) = 0 \quad \text{for each } \delta > 0.$$

In this case, F is uniquely determined, called the approximate differential of u at x_0, and denoted by $\nabla u(x_0)$.

b) We denote the set of approximate differentiability points of u by Ω_d.

Definition 3.2.2. *A function $u : \Omega \to \mathbb{R}^n$ is said to be injective a.e. in a subset A of Ω if there exists an $\mathscr{L}^n$-null subset N of A such that $u|_{A\setminus N}$ is injective.*

Next, we define the *geometric image* of a measurable set $A \subset \Omega$ under an approximately differentiable map $u : \Omega \to \mathbb{R}^n$. This was defined as $u(A \cap \Omega_d)$ by Müller and Spector [123]; however, we will use the following definition, which is an adaptation of that of Conti and De Lellis [31].

Definition 3.2.3. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and suppose that $\det Du \neq 0$ a.e. in Ω. Define Ω_0 as the set of $x \in \Omega$ for which the following are satisfied:*

1. *u is approximately differentiable at x and $\det \nabla u(x) \neq 0$; and*
2. *there exist $w \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at x such that $u|_K = w|_K$ and $\nabla u|_K = Dw|_k$.*

For any measurable set A of Ω, we define the geometric image of A under u as $u(A \cap \Omega_0)$, and we denote it by $\operatorname{im_G}(u, A)$.

From now on, we will use Du for both, the distributional derivative of u and for the approximate differential, ∇u, of u. Note, that if u is Sobolev, Du also denotes the weak derivative.

Given a measurable $u : \Omega \to \mathbb{R}^n$ that is approximately differentiable a.e., for any $A \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we denote by $\mathcal{N}_A(y)$ the number of $x \in \Omega_0 \cap A$ such that $u(x) = y$. We will use the following version of Federer's [59] area formula, the formulation of which is taken from [123, Proposition 2.6] (see also [72]). In fact, they formulate the change of variables formula for Ω_d, but, since $\Omega_0 \subset \Omega_d$ and $|\Omega_d \setminus \Omega_0| = 0$ the formula is also true using Ω_0 instead of Ω_d.

Proposition 3.2.4. *Let $u : \Omega \to \mathbb{R}^n$ be measurable and approximately differentiable a.e. Then, for any measurable set $A \subset \Omega$ and any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_A \varphi(u(x))\, |\det Du(x)|\, dx = \int_{\mathbb{R}^n} \varphi(y)\, \mathcal{N}_A(y)\, dy,$$

whenever either integral exists. Moreover, given $\psi : A \to \mathbb{R}$ measurable, the function $\bar{\psi} : u(\Omega_0 \cap A) \to \mathbb{R}$ defined by

$$\bar{\psi}(y) := \sum_{\substack{x \in \Omega_0 \cap A \\ u(x) = y}} \psi(x)$$

is measurable and satisfies

$$\int_A \psi(x)\, \varphi(u(x))\, |\det Du(x)|\, dx = \int_{u(\Omega_0 \cap A)} \bar{\psi}(y)\, \varphi(y)\, dy,$$

whenever the integral of the left-hand side exists.

We will use the topological degree for continuous functions [45, 61]: if $U \subset \mathbb{R}^n$ is a bounded open set, $u : \overline{U} \to \mathbb{R}^n$ is continuous and $y \in \mathbb{R}^n \setminus u(\partial U)$, we denote by $\deg(u, U, y)$ the degree of u in U at y. If $u : \partial U \to \mathbb{R}^n$ is continuous, its degree $\deg(u, U, \cdot)$ is defined as the degree of any continuous extension $\overline{u} : \overline{U} \to \mathbb{R}^n$, which exists thanks to Tietze's theorem and does not depend on the extension due to the homotopy-invariance of the degree (e.g. [45, Theorem 3.1.(d6)], [61, Theorem 2.4.]). If $u \in W^{1,p}(\partial U, \mathbb{R}^n)$ with $p > n - 1$, by Morrey's embedding, u has a continuous representative. We define the degree of u in U, written $\deg(u,U,\cdot)$, as the degree of its continuous representative.

Next, we define the topological image and the condition INV. The concept of topological image was introduced by Šverák [156].

Definition 3.2.5. *Let $p > n-1$ and let $U \subset\subset \mathbb{R}^n$ be a nonempty open set with a C^1 boundary. If $u \in W^{1,p}(\partial U, \mathbb{R}^n)$, we define $\operatorname{im}_{\mathrm{T}}(u,U)$, the topological image of U under u, as the set of $y \in \mathbb{R}^n \setminus u(\partial U)$ such that* $\deg(u,U,y) \neq 0$.

Thanks to the continuity of the topological degree for continuous functions we have that the topological image is an open set.

Now, we define condition INV, due to [123].

Definition 3.2.6. *Let $u \in W^{1,p}(\Omega,\mathbb{R}^n)$ with $p > n-1$. We say that u satisfies condition INV provided that for every $x_0 \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$, the following conditions hold:*

a) $u(x) \in \operatorname{im}_{\mathrm{T}}(u, B(x_0,r))$ *for a.e.* $x \in B(x_0,r)$.

b) $u(x) \notin \operatorname{im}_{\mathrm{T}}(u, B(x_0,r))$ *for a.e.* $x \in \Omega \setminus B(x_0,r)$.

Definition 3.2.7. *Given an open set U compactly contained in Ω with a C^2 boundary, define the function $d_U : \Omega \to \mathbb{R}$ as*

$$d_U(x) := \begin{cases} \operatorname{dist}(x,\partial U) & \text{if } x \in U \\ 0 & \text{if } x \in \partial U \\ -\operatorname{dist}(x,\partial U) & \text{if } x \in \Omega \setminus \overline{U}. \end{cases}$$

For $t > 0$ small enough, define the open set

$$U_t := \{x \in \Omega : d_U(x) > t\}.$$

Now we define the class of good open sets where we will work.

Definition 3.2.8. *Let $p > n-1$ and $u \in W^{1,p}(\Omega,\mathbb{R}^n)$. We define $\mathcal{U}_u$ as the class of nonempty open sets U that are compactly contained in Ω with a C^2 boundary and that satisfy the following conditions:*

1. $u|_{\partial U} \in W^{1,p}(\partial U, \mathbb{R}^n)$, *and* $(\operatorname{cof} Du)|_{\partial U} \in L^1(\partial U, \mathbb{R}^{n\times n})$.
2. $\lim_{\varepsilon\to 0} \fint_0^\varepsilon \left| \int_{\partial U_t} |\operatorname{cof} Du(x)| \, d\mathcal{H}^{n-1}(x) - \int_{\partial U} |\operatorname{cof} Du| \, d\mathcal{H}^{n-1} \right| dt = 0.$
3. $\mathcal{H}^{n-1}(\partial U \setminus \Omega_0) = 0$ *where Ω_0 is the set of Definition 3.2.3, and* $D(u|_{\partial U})(x) = Du(x)|_{T_x \partial U}$, *for $\mathcal{H}^{n-1}$- almost every $x \in \partial U$. Here $T_x\partial U$ is the linear tangent space of ∂U at x and $D(u|_{\partial U})(x)$ the tangential derivative of $u|_{\partial U}$ at x.*
4. *For every $\phi \in C^1(\Omega)$ and $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with* $(\operatorname{adj} Du)(g\circ u) \in L^1_{loc}(\Omega,\mathbb{R}^n)$,

$$\lim_{\varepsilon\to 0} \fint_0^\varepsilon \left| \int_{\partial U_t} \phi(x) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x)) \, d\mathcal{H}^{n-1}(x) - \int_{\partial U} \phi(x) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu(x)) \, d\mathcal{H}^{n-1}(x) \right| dt = 0.$$

where ν_t denotes the unit outward normal to U_t for each $t \in (0,\varepsilon)$, and ν is the unit outward normal to U.

The following result [78, Lemma 2.16.] guarantees that there are enough sets in $\mathcal{U}_u$.

Lemma 3.2.9. *[78, Lemma 2.16.] Suppose $u \in W^{1,p}(\Omega)$ with $p > n-1$ and let $U \subset \Omega$ be a nonempty open set with a C^2 boundary. Let $\delta > 0$ and d_U and U_t be defined as in Definition 3.2.7. Then $U_t \in \mathcal{U}_u$ for a.e. $t \in (-\delta, \delta)$.*

Definition 3.2.10. *If $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, we define $\operatorname{im}_{\mathrm{T}}(u,\Omega) = \bigcup_{U \in U_u} \operatorname{im}_{\mathrm{T}}(u,U)$.*

Note that $\operatorname{im}_{\mathrm{T}}(u,\Omega)$, as a union of open sets, is an open set. Moreover

$$\operatorname{im}_{\mathrm{T}}(u,\Omega) = \bigcup_{i\in\mathbb{N}} \operatorname{im}_{\mathrm{T}}(u,U_i)$$

for every family $\{U_i\}_{i\in\mathbb{N}} \subset \mathcal{U}_u$ such that $\Omega = \bigcup_{i\in\mathbb{N}} U_i$.

Given a function $u \in W^{1,p}(\Omega,\mathbb{R}^n)$ with $p > n-1$, we define $\mathcal{U}_u^{\mathrm{in}}$ as the the class of $U \in \mathcal{U}_u$ such that u is injective a.e. in U.

The following result, which is a consequence of [16, Corollary 4.7], shows, together with Lemma 3.2.9, that there are enough sets in $\mathcal{U}_u^{\mathrm{in}}$.

Proposition 3.2.11. *Let $u \in \mathcal{A}_p(\Omega)$. Then, for a.e. $x \in \Omega$ there exists $r > 0$ such that u is injective a.e. in $B(x,r)$, and $U \in \mathcal{U}_u^{in}$ for any $U \in \mathcal{U}_u$ such that $U \subset B(x,r)$.*

If $U \in \mathcal{U}_u^{\mathrm{in}}$ then u is injective in $U \cap \Omega_0$ [77, Lemma 3]. Therefore $u : U \cap \Omega_0 \to \operatorname{im}_{\mathrm{G}}(u,U)$ is a bijection. Moreover, thanks to [16, Theorem 4.1.] we have

$$|\operatorname{im}_{\mathrm{T}}(u,U) \setminus \operatorname{im}_{\mathrm{G}}(u,U)| = |\operatorname{im}_{\mathrm{G}}(u,U) \setminus \operatorname{im}_{\mathrm{T}}(u,U)| = 0$$

and, hence the next definition of local inverse of a function in the class $\mathcal{A}_p$, is well defined.

Definition 3.2.12. *Let $u \in \mathcal{A}_p(\Omega)$ and $U \in \mathcal{U}_u^{in}$. The inverse $(u|_U)^{-1} : \operatorname{im}_{\mathrm{T}}(u,U) \to \mathbb{R}^n$ is defined a.e. as $(u|_U)^{-1}(y) = x$, for each $y \in \operatorname{im}_{\mathrm{G}}(u,U)$, and where $x \in U \cap \Omega_0$ satisfies $u(x) = y$.*

By [16, Proposition 5.3.] we have

$$(u|_U)^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(u,U),\mathbb{R}^n) \quad \text{and} \quad D(u|_U)^{-1} = \left(Du \circ (u|_U)^{-1}\right)^{-1} \text{ a.e.}$$

The functional $\mathcal{E}$ defined below was introduced in [76] to measure the creation of new surface of a deformation. We are only interested in the case $\mathcal{E}(u) = 0$, i.e., when u does not create new surface.

Definition 3.2.13. *Let $u : \Omega \to \mathbb{R}^n$ be measurable and approximately differentiable a.e. Suppose that $\det Du \in L^1_{loc}(\Omega)$, $\operatorname{cof} Du \in L^1(\Omega, \mathbb{R}^{n\times n})$. For every $f \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, define*

$$\mathcal{E}_\Omega(u,f) := \int_\Omega \left[\operatorname{cof} Du(x) \cdot Df(x,u(x)) + \det Du(x) \operatorname{div} f(x,u(x))\right] dx \tag{3.2.1}$$

and

$$\mathcal{E}_\Omega(u) := \sup\left\{\mathcal{E}_\Omega(u,f) : f \in C^1_c\left(\Omega \times \mathbb{R}^n, \mathbb{R}^n\right), \|f\|_\infty \leqslant 1\right\}.$$

In equation (3.2.1), $Df(x,y)$ *denotes the derivative of* $f(\cdot,y)$ *evaluated at* x, *while* $\operatorname{div} f(x,y)$ *is the divergence of* $f(x,\cdot)$ *evaluated at* y.

For $\phi \in C^1(\Omega)$ *and* $g \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$ *define*

$$\mathscr{E}'_\Omega(u,\phi,g) := \int_\Omega \left[\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi(x)) + \det Du(x)\phi(x)\operatorname{div} g(u(x))\right] dx.$$

Clearly, if $\phi \in C^1_c(\Omega)$, and $g \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$ and we define $f \in C^1_c(\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ as $f(x,y) = \phi(x)g(y)$ then $\mathscr{E}_\Omega(u,f) = \mathscr{E}'_\Omega(u,\phi,g)$.

We will use in this chapter that, when $u \in W^{1,p}$ with $p > n-1$, $\mathscr{E}_\Omega(u) = 0$ is equivalent to $\mathscr{E}'_\Omega(u,\phi,g) = 0$, for all $\phi \in C^1_c(\Omega)$ and $g \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$. This can be shown by using the density in $C^1_c(\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ of sums of functions of separate variables (see e.g. [107, Corollary 1.6.5.]). Now we present the class of functions with which we will work in the rest of the chapter.

Definition 3.2.14. *For each* $p > n-1$ *and* $q \geqslant 1$, *we define* $\mathscr{A}_{p,q}(\Omega)$ *as the set of* $u \in W^{1,p}(\Omega,\mathbb{R}^n)$, *such that* $\det Du \in L^1_{loc}(\Omega)$, $\operatorname{cof} Du \in L^q(\Omega)$, $\det Du > 0$ *a.e. and* $\mathscr{E}_\Omega(u) = 0$. *We define* $\mathscr{A}_p(\Omega) = \mathscr{A}_{p,1}(\Omega)$. *We denote by* $\mathscr{A}^1_p(\Omega)$ *the set of functions* $u \in \mathscr{A}_p(\Omega)$ *that satisfy* $\det Du = 1$ *a.e.*

Observe that $u \in W^{1,p}$ implies $\operatorname{cof} Du \in L^{\frac{p}{n-1}}$, so $\mathscr{A}_p(\Omega) = \mathscr{A}_{p,t}(\Omega)$ for $t \in [1,\frac{p}{n-1}]$. Moreover, thanks to the result of [122] we have that if $u \in W^{1,p}$ satisfies $\operatorname{cof} Du \in L^q$ and $\det Du > 0$ a.e. with $p > n-1$ and $q \geqslant \frac{n}{n-1}$ then $u \in \mathscr{A}_{p,q}$.

Next, we define the notions of polyconvex and quasiconvex.

Let τ be the number of minors of $\mathbb{R}^{n\times n}$. We say that $f : \mathbb{R}^{n\times n} \to \mathbb{R}\cup\{\infty\}$ is polyconvex if there is a convex function $g : \mathbb{R}^\tau \to \mathbb{R}\cup\{\infty\}$ such that $f(F) = g(M(F))$, where $M(F)$ denotes the vector in $\mathbb{R}^\tau$ of all minors of F.

We say that $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is quasiconvex if for each open bounded subset Ω of $\mathbb{R}^n$, each $A \in \mathbb{R}^{n\times n}$ and any $\varphi : \Omega \to \mathbb{R}^n$ Lipschitz with compact support in Ω, the Jensen type inequality

$$f(A) \leqslant \frac{1}{|\Omega|}\int_\Omega f(A + D\varphi(x))\,dx$$

holds.

Next, for the convenience of the reader, we enunciate some results that we will use in this chapter. The following result is a weaker version of [33, Lemma A.1.].

Lemma 3.2.15. *Let* $\psi \in W^{1,\infty}(B(0,1),\overline{B}(0,1))$, $f \in L^1(B(0,2),\mathbb{R})$. *Then the map* $(x,a_0) \to f(a_0 + \psi(x-a_0))$ *is* $\mathscr{L}^{2n}$*-measurable and for almost all* $a_0 \in B(0,1)$ *the function*

$$x \to f(a_0 + \psi(x-a_0))$$

is in $L^1(B(a_0,1))$.

In this chapter we will use two different chain rules. The first one is as follows.

Lemma 3.2.16. *[33, Lemma A.2.] Let* $\psi \in W^{1,\infty}(B(0,1),\overline{B}(0,1))$, $u \in W^{1,1}(B(0,2))$. *Then for almost all* $a_0 \in B(0,1)$ *the function* $w(x) = u(a_0 + \psi(x-a_0))$ *belongs to* $W^{1,1}(B(a_0,1))$ *with*

$$Dw(x) = Du(a_0 + \psi(x-a_0))D\psi(x-a_0).$$

If, in addition, $\psi(x) = x$ *on* $\partial B(0,1)$ *then* $w = u$ *on* $\partial B(a_0,1)$ *in the sense of traces.*

The other chain rule that we will use is in the case $\phi \in W^{1,\infty}(B)$ and $\rho \in W^{1,1}(B,B)$. Then, thanks to [164, Theorem 2.1.11.] we have $\phi \circ \rho \in W^{1,1}(B)$ and $D(\phi \circ \rho) = D\phi(\rho)D\rho$.

Theorem 3.2.17. *[16, Theorem 4.1.] Let $u \in \mathcal{A}_p(\Omega)$, $p > n-1$. Then for all $U \in \mathcal{U}_u$,*

$$\deg(u,U,\cdot) = \mathcal{N}_U \quad a.e.$$

Theorem 3.2.18. *[79, Theorem 3.3.] Let $u \in \mathcal{A}_p(\Omega)$, with $p > n-1$, satisfy INV. Then $u^{-1} \in W^{1,1}(\mathrm{im}_{\mathrm{T}}(u,\Omega))$ and $Du^{-1}(y) = Du(u^{-1}(y))^{-1}$ a.e. $y \in \mathrm{im}_{\mathrm{T}}(u,\Omega)$.*

The following proposition is a result of compacity of $\vec{n}_j$ and of lower semicontinuity for functionals of type $\int_{\mathrm{im}_{\mathrm{T}}(u,\Omega)} W(D\vec{n})dx$.

Proposition 3.2.19. *[16, Proposition 7.1.] Let $r > 1$, $p > n-1$ and $d \in \mathbb{N}$ and K a compact subset of $\mathbb{R}^d$. For each $j \in \mathbb{N}$, let $u, u_j \in \mathcal{A}_p(\Omega)$ and $\vec{n}_j \in W^{1,r}(\mathrm{im}_{\mathrm{T}}(u_j,\Omega),K)$ be such that*

$$u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega) \text{ as } j \to \infty \quad \text{and} \quad \sup_{j\in\mathbb{N}} \left[\|\vec{n}_j\|_{L^1(\mathrm{im}_{\mathrm{T}}(u_j,\Omega),\mathbb{R}^d)} + \|D\vec{n}_j\|_{L^r(\mathrm{im}_{\mathrm{T}}(u_j,\Omega),\mathbb{R}^{d\times n})} \right] < \infty.$$

Assume that $\det Du \in L^1(\Omega)$ *and let $W : \mathbb{R}^{d\times n} \to [0,\infty)$ be a quasiconvex function for which there is a $c > 0$ with*

$$W(F) \leq c(1+|F|^r), \quad F \in \mathbb{R}^{d\times n}.$$

Then there is $\vec{n} \in W^{1,r}(\mathrm{im}_{\mathrm{T}}(u,\Omega),K)$ such that

$$\int_{\mathrm{im}_{\mathrm{T}}(u,\Omega)} W(D\vec{n}(y))dy \leq \liminf_{j\to\infty} \int_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)} W(D\vec{n}_j(y))dy$$

and, for a subsequence,

$$\chi_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)}\vec{n}_j \to \chi_{\mathrm{im}_{\mathrm{T}}(u,\Omega)}\vec{n} \text{ in } L^r \text{ and a.e.},$$
$$\chi_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)}D\vec{n}_j \rightharpoonup \chi_{\mathrm{im}_{\mathrm{T}}(u,\Omega)}D\vec{n} \text{ in } L^r \text{ as } j \to \infty.$$

The following result is about the semicontinuity of $\int_\Omega W(Du, \vec{n} \circ u)dx$ and is a particular case of [16, Proposition 7.8.].

Proposition 3.2.20. *Let $p > n-1$ and $d \in \mathbb{N}$. For each $j \in \mathbb{N}$, let $u, u_j \in \mathcal{A}_p(\Omega)$ and let*

$$\vec{n} : \mathrm{im}_{\mathrm{T}}(u,\Omega) \to \mathbb{S}^{n-1} \quad \text{and} \quad \vec{n}_j : \mathrm{im}_{\mathrm{T}}(u_j,\Omega) \to \mathbb{S}^{n-1}$$

be measurable such that

$$u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \chi_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)}\vec{n}_j \to \chi_{\mathrm{im}_{\mathrm{T}}(u,\Omega)}\vec{n} \quad \text{a.e. as } j \to \infty.$$

Suppose, in addition, that there is a Borel function $\theta : (0,\infty) \to [0,\infty)$ such that

$$\lim_{t\to 0} \theta(t) = \infty$$

and

$$\sup_{j\in\mathbb{N}} \int_\Omega \theta(\det Du_j(x))dx < \infty.$$

Let $W : \mathbb{R}^{n\times n}_+ \times \mathbb{S}^{n-1} \to [0,\infty)$ be a continuous function such that $W(\cdot,\vec{m})$ is polyconvex for all $\vec{m} \in \mathbb{S}^{n-1}$. Then

$$\int_\Omega W(Du(x),\vec{n}(u(x)))dx \leq \liminf_{j\to\infty} \int_\Omega W(Du_j(x),\vec{n}_j(u_j(x)))dx.$$

3.3 Some results for functions in the class $\mathscr{A}_p$

In this section we provide some auxiliary results for functions in $\mathscr{A}_p$ that will be used in Section 3.5 and cannot be found elsewhere. We think that some of them have interest by themselves.

The next result is a representation of $\mathscr{E}'(u,\phi,g)$ as a surface integral. The proof is similar to that of [76, Theorem 2].

Lemma 3.3.1. *Let* $\Omega \subset \mathbb{R}^n$, $u \in \mathscr{A}_p(\Omega)$, $U \in \mathscr{U}_u$, $\phi \in C^1(\Omega)$ *and* $g \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$. *Then*

$$\mathscr{E}'_U(u,\phi,g) = \int_{\partial U} \phi(x)\operatorname{cof} Du(x)\cdot(g(u(x))\otimes \nu(x))\,d\mathscr{H}^{n-1}(x).$$

Proof. Let $\varepsilon > 0$ and set $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\varphi(x) = 0$ for $x \leqslant 0$, $\varphi(x) = 1$ for $x \geqslant 1$, $0 \leqslant \varphi'(x) < 1+3\varepsilon$ for all $x \in \mathbb{R}$, and $\varphi'(x) = 1$ for $\varepsilon < x < 1-\varepsilon$. Recall the function d_U of Definition 3.2.7 and for each $j \in \mathbb{N}$, define $\eta_j \in C^1_0(U)$ as $\eta_j(x) = \varphi(j d_U(x))$. Hence η_j satisfies $0 \leqslant \eta_j(x) \leqslant 1$ for $x \in U$ and $\eta_j(x) = 1$ in $U_{j^{-1}}$. Set $\phi_j(x) = \eta_j(x)\phi(x)$, so $\phi_j \in C^1_0(U)$. Then $\phi_j \to \phi$ a.e. and $\eta_j \to \eta$ a.e., so using the Dominated Convergence Theorem we get

$$\lim_{j\to\infty} \int_U \det Du(x)\phi_j(x)\operatorname{div} g(u(x))\,dx = \int_U \det Du(x)\phi(x)\operatorname{div} g(u(x))\,dx \tag{3.3.1}$$

and

$$\lim_{j\to\infty} \int_U \eta_j(x)\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi(x))\,dx = \int_U \operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi(x))\,dx. \tag{3.3.2}$$

We also denote by ϕ_j the extension of ϕ_j to Ω by zero, so $\phi_j \in C^1_c(\Omega)$ as well. So, using

$$\begin{aligned} 0 = \mathscr{E}'_U(u,\phi_j,g) &= \int_\Omega \left[\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi_j(x)) + \det Du(x)\phi_j(x)\operatorname{div} g(u(x))\right] dx \\ &= \int_U \left[\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi_j(x)) + \det Du(x)\phi_j(x)\operatorname{div} g(u(x))\right] dx, \end{aligned}$$

we get

$$\begin{aligned} &\int_U \det Du(x)\phi_j(x)\operatorname{div} g(u(x))\,dx = -\int_U \operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi_j(x))\,dx \\ &= -\int_U \phi(x)\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\eta_j(x))\,dx - \int_U \eta_j(x)\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\phi(x))\,dx. \end{aligned}$$

Passing to the limit, using (3.3.1), (3.3.2) and the Coarea formula we obtain

$$\begin{aligned} \mathscr{E}'_U(u,\phi,g) &= -\lim_{j\to\infty}\int_U \phi(x)\operatorname{cof} Du(x)\cdot(g(u(x))\otimes D\eta_j(x))\,dx \\ &= -\lim_{j\to\infty}\int_{U\setminus U_{j^{-1}}} \phi(x)\,j\varphi'(j d_U(x))\operatorname{cof} Du(x)\cdot(g(u(x))\otimes Dd_U(x))\,dx \\ &= \lim_{j\to\infty} ⨍_0^{j^{-1}} \int_{\partial U_t} \phi(x)\varphi'(j d_U(x))\operatorname{cof} Du(x)\cdot(g(u(x))\otimes \nu_t(x))\,d\mathscr{H}^{n-1}(x)\,dt. \end{aligned}$$

In the last equality we have used $Dd_U = -\nu_t$ on ∂U_t.

Thanks to $\varphi' = 1$ in $[\varepsilon, 1-\varepsilon]$ we have $\varphi'(jd_U(x)) = 1$ for $x \in \partial U_t$ and $\varepsilon j^{-1} \leqslant t \leqslant (1-\varepsilon) j^{-1}$. Therefore,

$$
\begin{aligned}
&\fint_0^{j^{-1}} \int_{\partial U_t} \phi(x)\varphi'(jd_U(x)) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x))\, d\mathcal{H}^{n-1}(x)\, dt \\
&= \fint_0^{j^{-1}} \int_{\partial U_t} \phi(x) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x))\, d\mathcal{H}^{n-1}(x)\, dt \\
&+ j \left[\int_0^{\varepsilon j^{-1}} + \int_{(1-\varepsilon) j^{-1}}^{j^{-1}} \right] \int_{\partial U_t} \phi(x)(\varphi'(jd_U(x)) - 1) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x))\, d\mathcal{H}^{n-1}(x)\, dt.
\end{aligned}
$$

But

$$
\fint_0^{j^{-1}} \int_{\partial U_t} \phi(x) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x))\, d\mathcal{H}^{n-1}(x)\, dt
$$

converges when $j \to \infty$ to

$$
\int_{\partial U} \phi(x) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu(x))\, d\mathcal{H}^{n-1}(x)\, dt,
$$

and

$$
\begin{aligned}
&\left| j \left[\int_0^{\varepsilon j^{-1}} + \int_{(1-\varepsilon) j^{-1}}^{j^{-1}} \right] \int_{\partial U_t} \phi(x)(\varphi'(jd_U(x)) - 1) \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes \nu_t(x))\, d\mathcal{H}^{n-1}(x)\, dt \right| \\
&\leqslant j \left[\int_0^{\varepsilon j^{-1}} + \int_{(1-\varepsilon) j^{-1}}^{j^{-1}} \right] \int_{\partial U_t} |\phi(x)| |\operatorname{cof} Du(x)| |g(u(x))|\, d\mathcal{H}^{n-1}(x)\, dt \\
&\leqslant \|\phi\|_{L^\infty(U)} \|g\|_{L^\infty} j \left[\int_0^{\varepsilon j^{-1}} + \int_{(1-\varepsilon) j^{-1}}^{j^{-1}} \right] \int_{\partial U_t} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)\, dt.
\end{aligned}
$$

Let $a > 0$, then, thanks to Definition 3.2.8 we have

$$
\lim_{j\to\infty} aj \int_0^{(aj)^{-1}} \int_{\partial U_t} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)\, dt = \int_{\partial U} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x).
$$

Therefore,

$$
\begin{aligned}
&\lim_{j\to\infty} j \int_{(1-\varepsilon) j^{-1}}^{j^{-1}} \int_{\partial U_t} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)\, dt = \int_{\partial U} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x) \\
&- \lim_{j\to\infty} j \int_0^{(1-\varepsilon) j^{-1}} \int_{\partial U_t} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)\, dt = \varepsilon \int_{\partial U} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)
\end{aligned}
$$

and

$$
\lim_{j\to\infty} j \int_0^{\varepsilon j^{-1}} \int_{\partial U_t} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x)\, dt = \varepsilon \int_{\partial U} |\operatorname{cof} Du(x)|\, d\mathcal{H}^{n-1}(x).
$$

Therefore,

$$\begin{aligned}&\left|\mathscr{E}'_U(u,\phi,g)-\int_{\partial U}\phi(x)\operatorname{cof}Du(x)\cdot(g(u(x))\otimes\nu(x))d\mathscr{H}^{n-1}(x)\right|\\&\leqslant 2\varepsilon\|\phi\|_{L^\infty(U)}\|g\|_{L^\infty}\int_{\partial U}|\operatorname{cof}Du(x)|d\mathscr{H}^{n-1}(x).\end{aligned}$$

Hence, letting ε go to zero we have

$$\mathscr{E}'_U(u,\phi,g)=\int_{\partial U}\phi(x)\operatorname{cof}Du(x)\cdot(g(u(x))\otimes\nu(x))d\mathscr{H}^{n-1}(x).$$

□

In the next lemma we show that if we paste two functions in the class $\mathscr{A}_p$ that coincide in a neighborhood of a sphere, then the resulting function is also in the class $\mathscr{A}_p$.

Lemma 3.3.2. *Let $\Omega\subset\mathbb{R}^n$, $B\subset\subset\Omega$, $B'\subset\subset B$, $u\in\mathscr{A}_{p,q}(\Omega)$, $v\in\mathscr{A}_{p,q}(B)$ such that B and B' are open, $u(x)=v(x)$ for $x\in B\setminus B'$ and $|v(B)\cap u(\Omega\setminus B)|=0$. Then the function*

$$w(x):=\begin{cases}v(x) & \text{for } x\in B',\\ u(x) & \text{for } x\in\Omega\setminus B',\end{cases}$$

is in $\mathscr{A}_{p,q}(\Omega)$. Moreover, if $u\in\mathscr{A}^1_p(\Omega)$ and $v\in\mathscr{A}^1_p(B)$, then we also have $w\in\mathscr{A}^1_p(\Omega)$.

Proof. All the conditions in the definition of $\mathscr{A}_{p,q}$ are immediate to check except $\mathscr{E}(w)=0$, so we only have to prove that.

Let $U\subset\subset B$ be such that $U\in\mathscr{U}_u\cap\mathscr{U}_v$ and $B'\subset\subset U$. Let $\phi\in C^1_c(\Omega)$ and $g\in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$. Thanks to Lemma 3.3.1 we have

$$\begin{aligned}\mathscr{E}'_U(w,\phi,g)=\mathscr{E}'_U(v,\phi,g)&=\int_{\partial U}\phi(x)\operatorname{cof}Dv(x)\cdot(g(v(x))\otimes\nu(x))d\mathscr{H}^{n-1}(x)\\&=\int_{\partial U}\phi(x)\operatorname{cof}Du(x)\cdot(g(u(x))\otimes\nu(x))d\mathscr{H}^{n-1}(x).\end{aligned}$$

The last equality arises from the equality $u(x)=v(x)$ for $x\in B\setminus B'$ and $\partial U\subset B\setminus B'$. On the other hand, using also that $\mathscr{E}'_\Omega(u,\phi,g)=0$, we get

$$\mathscr{E}'_{\Omega\setminus U}(w,\phi,g)=\mathscr{E}'_\Omega(u,\phi,g)-\mathscr{E}'_U(w,\phi,g)=-\int_{\partial U}\phi(x)\operatorname{cof}Du(x)\cdot(g(u(x))\otimes\nu(x))d\mathscr{H}^{n-1}(x).$$

Therefore,

$$\mathscr{E}'_\Omega(w,\phi,g)=\mathscr{E}'_{\Omega\setminus U}(w,\phi,g)+\mathscr{E}'_U(w,\phi,g)=0.$$

□

In the next lemma we see that the functional $\mathscr{E}'_\Omega(u,\phi,g)$ is also zero for ϕ in the correct Sobolev space and $u\in\mathscr{A}_{p,q}$.

Lemma 3.3.3. *Let $p > n-1$ and $q > 1$, $\Omega \subset \mathbb{R}^n$, $\Omega' \subset\subset \Omega$ with Lipschitz boundary, $u \in \mathscr{A}_{p,q}(\Omega)$, $g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, $\phi \in W^{1,q'}(\Omega) \cap L^\infty(\Omega)$ with $\phi|_{\Omega\setminus\Omega'} = 0$. Then $\mathscr{E}'_\Omega(u,\phi,g) = 0$.*

Proof. Let $\phi_j \in C_c^1(\Omega)$ be such that $\phi_j|_{\Omega\setminus\Omega'} = 0$, $\phi_j \underset{j\to\infty}{\to} \phi$ in $W^{1,q'}(\Omega)$ and $\phi_j \underset{j\to\infty}{\overset{*}{\rightharpoonup}} \phi$ in $L^\infty(\Omega)$. Then, by the dominated convergence theorem we have that

$$\lim_{j\to\infty} \int_\Omega \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes D\phi_j(x))\, dx = \int_\Omega \operatorname{cof} Du(x) \cdot (g(u(x)) \otimes D\phi(x))\, dx$$

and

$$\lim_{j\to\infty} \int_\Omega \det Du(x) \phi_j(x) \operatorname{div} g(u(x))\, dx = \int_\Omega \det Du(x) \phi(x) \operatorname{div} g(u(x))\, dx.$$

Then

$$\mathscr{E}'_\Omega(u,\phi,g) = \lim_{j\to\infty} \mathscr{E}'_\Omega(u,\phi_j,g) = 0.$$

□

In the rest of the chapter we will continuously use the following identities for $\rho \in \mathscr{A}_p$ and $y \in \operatorname{im}_T(\rho, U)$ with $U \in \mathscr{U}_\rho^{\text{in}}$:

- $\det D\rho(\rho^{-1}(y)) = \frac{1}{\det D\rho^{-1}(y)}$,
- $\operatorname{cof} D\rho(\rho^{-1}(y)) = \frac{D\rho^{-1}(y)^T}{\det D\rho^{-1}(y)}$.

Next, we prove that the composition of a function in the class $\mathscr{A}_{p,q}$ with a Lipschitz function satisfying some conditions, is still in the class $\mathscr{A}_{p,q}$.

Lemma 3.3.4. *Let $u \in \mathscr{A}_{p,q}(\Omega)$, $B \subset\subset \Omega$ a ball, $\rho : B \to \overline{B}$ Lipschitz such that $\rho(x) = x$ for $x \in \partial B$, $\det D\rho > 0$ a.e. and $\int_\Omega \left(\frac{1}{\det D\rho}\right)^{q'-1} dx < \infty$,*

$$z(x) := \begin{cases} u(\rho(x)) & \text{for } x \in B, \\ u(x) & \text{for } x \in \Omega \setminus B. \end{cases}$$

Then ρ is invertible a.e. Moreover, if we also have $\rho^{-1} \in W^{1,1}(B)$, $z \in W^{1,p}(\Omega)$, $Dz = Du(\rho)D\rho$ in B, $\det Dz \in L^1_{loc}(\Omega)$ and that $\operatorname{cof} Dz \in L^q(\Omega)$; then $z \in \mathscr{A}_{p,q}(\Omega)$. If in addition, $u \in \mathscr{A}_p^1(\Omega)$ and $\det D\rho = 1$ a.e., we also have $z \in \mathscr{A}_p^1(\Omega)$.

Proof. The fact that ρ is invertible a.e. comes from [11]. By definition of $\mathscr{A}_{p,q}$, to prove $z \in \mathscr{A}_{p,q}(\Omega)$ we only have to show that $\mathscr{E}_\Omega(z) = 0$.

For $\phi \in C_c^1(\Omega)$ and $g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ we have

$$\mathscr{E}'_\Omega(z,\phi,g) = \mathscr{E}'_{\Omega\setminus B}(u,\phi,g) + \mathscr{E}'_B(u \circ \rho, \phi, g).$$

Therefore, using the change of variables formula we get

$$\begin{aligned}\mathscr{E}'_B(u\circ\rho,\phi,g) &= \int_B \big[\operatorname{cof}(Du(y)D\rho(\rho^{-1}(y)))\cdot(g(u(y))\otimes D\phi(\rho^{-1}(y)))\\ &\quad+\det(Du(y)D\rho(\rho^{-1}(y)))\phi(\rho^{-1}(y))\operatorname{div} g(u(y))\big]\frac{dy}{\det D\rho(\rho^{-1}(y))}\\ &= \int_B \big[\operatorname{cof}(Du(y))D\rho^{-1}(y)^T\cdot(g(u(y))\otimes D\phi(\rho^{-1}(y)))\\ &\quad+\det(Du(y))\phi(\rho^{-1}(y))\operatorname{div} g(u(y))\big]\,dy.\end{aligned}$$

Call

$$\tilde\phi(y)=\begin{cases}\phi(y) & \text{for } y\in\Omega\setminus B,\\ \phi(\rho^{-1}(y)) & \text{for } y\in B.\end{cases}$$

Since ϕ is Lipschitz and $\rho^{-1}\in W^{1,1}(B)$ we can use the chain rule [164, Theorem 2.1.11.] to get that $\tilde\phi\in W^{1,1}(B)$ and

$$D\tilde\phi(y)=D\phi(\rho^{-1}(y))D\rho^{-1}(y)\quad\text{for } y\in B.$$

Moreover, as $\rho|_{\partial B}=\mathrm{id}|_B$ then $\tilde\phi\in W^{1,1}(\Omega)$.

Let $\tilde B\subset\subset\Omega$ be with Lipschitz boundary such that $\operatorname{supp}\phi\subset\tilde B$. Clearly, $\tilde\phi|_{\Omega\setminus\tilde B}=0$ and $\tilde\phi\in L^\infty(\Omega)$, since $\phi\in C^1_c(\tilde B)$. We claim that $\tilde\phi\in W^{1,q'}(\tilde B)$, for which it is enough to see that $D\tilde\phi\in L^{q'}(\tilde B)$. In fact, as $\phi\in C^1_c(\Omega)$, its enough to check $\|D\tilde\phi\|_{L^{q'}(B)}<\infty$. Using that ρ and ϕ are Lipschitz and $\int_B\left(\frac{1}{\det D\rho}\right)^{q'-1}<\infty$ we get

$$\begin{aligned}\|D\tilde\phi\|^{q'}_{L^{q'}(B)} &= \int_B |D\phi(\rho^{-1}(x))|^{q'}|D\rho^{-1}(x)|^{q'}\,dx \lesssim \int_B |D\rho^{-1}(\rho(y))|^{q'}\det D\rho(y)\,dy\\ &= \int_B |\operatorname{cof} D\rho(y)|^{q'}(\det D\rho(y))^{1-q'}\,dy \lesssim \int_B(\det D\rho(y))^{1-q'}\,dy<\infty.\end{aligned}$$

Equality $\mathscr{E}'_\Omega(z,\phi,g)=\mathscr{E}'_\Omega(u,\tilde\phi,g)$ can be seen by change of variables with ρ. Thanks to Lemma 3.3.3 we have $\mathscr{E}'_\Omega(z,\phi,g)=0$. So, we have proved that $\mathscr{E}_\Omega(z)=0$ and that $z\in\mathscr{A}_{p,q}(\Omega)$.

If, in addition, $u\in\mathscr{A}^1_p(\Omega)$ and $\det D\rho=1$ a.e. then $u\in\mathscr{A}_{p,q}(\Omega)$ for $q=\frac{p}{n-1}$ and $\det Du=1$ a.e., so $z\in\mathscr{A}_{p,q}(\Omega)$, $\det Dz(x)=\det Du(\rho(x))\det D\rho(x)=1$ for a.e. $x\in B$ and $\det Dz(x)=\det Du(x)=1$ for a.e. $x\in\Omega\setminus B$. Therefore we get $z\in\mathscr{A}^1_p(\Omega)$. □

3.4 Invertibility in the class $\mathscr{A}_p$

In [16] they prove local invertibility for the class $\mathscr{A}_p$. In fact, with the same techniques it is possible to prove global invertibility, as we see in the next theorem, which generalizes [134, Theorem 3.7.] and shows that all the approaches to invertibility are equivalent in the class $\mathscr{A}_p$ with $p > n-1$. Approach *1)* below is based on injectivity a.e., approach *2)* on condition INV, approach *3)* on the degree, approach *4)* on the area formula and approach *5)* on the number of preimages. We have grouped the 14 equivalent statements into these five categories for ease of reading. More of the history of these approaches can be found in the introduction of this chapter.

Theorem 3.4.1. *Let $u \in \mathscr{A}_p(\Omega)$. Then the following conditions are equivalent:*

1) a) There exists $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_u$ increasing such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$ and u is injective a.e. in U_j for all $j \in \mathbb{N}$.

b) u is injective a.e. in U for all $U \in \mathscr{U}_u$.

c) u is injective a.e. in Ω.

2) a) There exists $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_u$ increasing such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$ and $u|_{U_j}$ satisfies condition INV for all $j \in \mathbb{N}$.

b) $u|_U$ satisfies condition INV for all $U \in \mathscr{U}_u$.

c) u satisfies condition INV.

3) a) There exists $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_u$ increasing such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$ and $\deg\big(u, U_j, \cdot\big) \leqslant 1$ in $\mathbb{R}^n \setminus u(\partial U_j)$ for all $j \in \mathbb{N}$.

b) $\deg(u, U, \cdot) \leqslant 1$ in $\mathbb{R}^n \setminus u(\partial U)$, $\forall U \in \mathscr{U}_u$.

4) a) $\int_\Omega \det Du(x)\,dx \leqslant \mathscr{L}^n(\mathrm{im}_G(u,\Omega))$.

b) $\int_\Omega \det Du(x)\,dx = \mathscr{L}^n(\mathrm{im}_G(u,\Omega))$.

5) a) There exists $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_u$ increasing such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$ and $\mathscr{N}_{U_j} \leqslant 1$ a.e. for all $j \in \mathbb{N}$.

b) $\mathscr{N}_U \leqslant 1$ a.e. $\forall U \in \mathscr{U}_u$.

c) $\mathscr{N}_\Omega \leqslant 1$ a.e.

d) $\mathscr{N}_\Omega \leqslant 1$.

If u satisfies these conditions, then $u^{-1} \in W^{1,1}(\mathrm{im_T}(u,\Omega), \mathbb{R}^n)$.

Proof. The implications *1c)⇒1b), 4b)⇒ 4a)* and *5d)⇒5c)⇒5b)* are clear.

1b)⇒1a), 2b)⇒2a), 3b)⇒3a) and *5b)⇒5a).* Clearly, it is enough to see that there exists an increasing sequence $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_u$ such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$, but this is guaranteed by Lemma 3.2.9.

1a)⇒1b) and *5a)⇒5b).* Given $U \in \mathscr{U}_u$, take $j_0 \in \mathbb{N}$ such that $U \subset U_{j_0}$. Then the implications are clear.

1a)⇒1c). Let $N_j \subset U_j$ be such that u is injective in $U_j \setminus N_j$ and $|N_j| = 0$. Then, if we denote $N = \bigcup_{j\in\mathbb{N}} N_j$, we obtain that u is injective in $\Omega \setminus N$ and $|N| = 0$. Therefore u is injective a.e. in Ω.

1a)⇔2a) and *1b)⇔2b)* are Lemma 5.1 of [16].

2b)⇔2c). It is a consequence of [78, Lemma 8.3.]

5b)⇒ 1b). Fix $U \in \mathscr{U}_u$ and let $N' \subset \mathbb{R}^n$ be such that $|N'| = 0$ and $\mathscr{N}_U(y) \leqslant 1$ for all $y \in \mathbb{R}^n \setminus N'$. Denote by N the set $\{x \in U \cap \Omega_0 : \exists x' \in U \cap \Omega_0,\, x' \neq x, \text{ and } u(x') = u(x)\}$. Since $|\Omega \setminus \Omega_0| = 0$ in order to prove *1b)* it is enough to see that $|N| = 0$. Using that $u|_{\Omega_0}$ also satisfies the condition N^{-1} (see [16, Lemma 2.8.c)]), from $|N'| = 0$ and $u(N) \subset N'$ we deduce that $|N| = 0$.

1c)⇒5d). Using [77, Lemma 3] we obtain that $u|_{\Omega_0}$ is injective. Therefore, for all $y \in \mathbb{R}^n$ we have that there exists at most one $x \in \Omega_0$ such that $u(x) = y$, i.e. $\mathscr{N}_\Omega(y) \leqslant 1$ and *5d)* is proved.

3a)⇔5a) and *3b)⇔5b)*. This is Theorem 3.2.17 and the fact that the degree is continuous.

For the following implications we will use Proposition 3.2.4 and that $\mathscr{N}_\Omega(y) \geqslant 1$ for $y \in u(\Omega_0)$ and $\mathscr{N}_\Omega(y) = 0$ for $y \in \mathbb{R}^n \setminus u(\Omega_0)$.

4a)⇒4b). We have

$$\int_\Omega \det Du(x)\,dx = \int_{\mathbb{R}^n} \mathscr{N}_\Omega(y)\,dy = \int_{u(\Omega_0)} \mathscr{N}_\Omega(y)\,dy \geqslant |u(\Omega_0)| = \mathscr{L}^n\left(\operatorname{im}_G(u,\Omega)\right).$$

Hence, applying *4a)* we get *4b)*.

5d)⇒4a). Using *5d)* we get

$$\int_\Omega \det Du(x)\,dx = \int_{\mathbb{R}^n} \mathscr{N}_\Omega(y)\,dy = \int_{u(\Omega_0)} \mathscr{N}_\Omega(y)\,dy \leqslant |u(\Omega_0)| = \mathscr{L}^n\left(\operatorname{im}_G(u,\Omega)\right).$$

4a)⇒5c). Let $A \subset \mathbb{R}^n$ be the set of $y \in \mathbb{R}^n$ such that $\mathscr{N}_\Omega(y) \geqslant 2$. It is clear that $A \subset u(\Omega_0)$. Then

$$\int_{\mathbb{R}^n} \mathscr{N}_\Omega(y)\,dy = \int_{u(\Omega_0)} \mathscr{N}_\Omega(y)\,dy \geqslant |u(\Omega_0) \setminus A| + 2|A| = \mathscr{L}^n\left(\operatorname{im}_G(u,\Omega)\right) + |A|.$$

On the other hand, using *4a)* we obtain

$$\int_{\mathbb{R}^n} \mathscr{N}_\Omega(y)\,dy = \int_\Omega \det Du(x)\,dx \leqslant \mathscr{L}^n\left(\operatorname{im}_G(u,\Omega)\right).$$

Therefore, $|A| = 0$ and *5c)* holds.

Finally, if u satisfies any of these conditions, using Theorem 3.2.18 we have

$$u^{-1} \in W^{1,1}(\operatorname{im}_T(u,\Omega),\mathbb{R}^n).$$

□

The next theorem extends the pioneering result of Ball [11] where he proved that if two functions $u, u_0 \in W^{1,p}$, with $p > n$, coincide in the border of Ω and u_0 is invertible and satisfy $\det Du_0 > 0$ a.e., then u is also invertible, which is a version for Sobolev maps of a classical result for C^1 maps. We extend this result to the class $\mathscr{A}_p$. However, in order to do so, we have to ask a little more that the traces being the same; we need that the functions coincide in a neighborhood of Ω. The necessity of this condition, that Šverák also uses in [156], arises from the fact that it is possible that Ω could be a bad set for the functions u and u_0 in the sense that $\Omega \notin \mathscr{U}_{\tilde u}$ or $\Omega \notin \mathscr{U}_{\tilde u_0}$, where $\tilde u$ and $\tilde u_0$ are extensions of u and u_0, respectively, to a neighborhood of $\overline{\Omega}$. If that is the case, the topological degree in Ω may not be defined and, if so, may not be monotone so it is possible that $\deg(u,\Omega,y) = \deg(u_0,\Omega,y) \neq 1$ for $y \in u(\Omega) \setminus u(\partial\Omega)$.

Theorem 3.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega' \subset\subset \Omega$, $p > n-1$ and $u_0, u \in \mathscr{A}_p(\Omega)$ such that u_0 satisfies the conditions of Theorem 3.4.1 and $u = u_0$ in $\Omega \setminus \Omega'$. Then u also satisfies the conditions of Theorem 3.4.1.*

Proof. Take a sequence $\{U_j\}_{j\in\mathbb{N}} \subset \mathscr{U}_{u_0} \cap \mathscr{U}_u$ such that $\bigcup_{j\in\mathbb{N}} U_j = \Omega$ and $\Omega' \subset U_j$ for all $j \in \mathbb{N}$, which exists thanks to Lemma 3.2.9. Thanks to part *3b)* of Theorem 3.4.1 we have $\deg\big(u_0, U_j, \cdot\big) \leqslant 1$ in $\mathbb{R}^n \setminus u_0(\partial U_j)$, hence, using $u = u_0$ in $\Omega \setminus \Omega'$, we get $\deg\big(u, U_j, \cdot\big) \leqslant 1$ in $\mathbb{R}^n \setminus u(\partial U_j)$. That proves that u satisfies *3a)* of Theorem 3.4.1 and, since all the conditions of Theorem 3.4.1 are equivalent, u satisfies all of them. $\square$

We finish this section by showing that the conditions of Theorem 3.4.1 are stable under weak convergence. Its proof is a direct consequence of [123, Lemma 3.3.], who showed the stability of condition INV.

Proposition 3.4.3. *Let $\{u_j\}_{j\in\mathbb{N}} \subset \mathscr{A}_p(\Omega)$ and $u \in \mathscr{A}_p(\Omega)$. Assume that $u_j \rightharpoonup u$ in $W^{1,p}$. If u_j satisfies the conditions of Theorem 3.4.1 for each $j \in \mathbb{N}$ then so does u.*

3.5 Relaxation of a model for nematic elastomers

In this section we deal with the relaxation of the energy of a model of nematic elastomers, investigated in [15], where the elastic behavior of the polymer chains is coupled to the orientation order for the nematic mesogens through the minimization a total free energy

$$\int_\Omega W_{\text{mec}}(Du(x),\vec{n}(u(x)))\,dx+\int_{\operatorname{im}_{\mathrm{T}}(u,\Omega)}|D\vec{n}(y)|^2\,dy,$$

where W_{mec} is continuous and has p-growth from below in the first variable. The function $\vec{n}$, which indicates the orientation of the mesogens, is defined in the deformed configuration and takes values in $\mathbb{S}^{n-1}$. The physical relevance of this model can be found in [15], [47] and [162]. In [16] they study this energy in the case that W_{mec} is polyconvex and u is in the class $\mathscr{A}_p$ with $p>n-1$. To deal with the composition $\vec{n}\circ u$, they work in the deformed configuration and use that the functions in $\mathscr{A}_p$ are invertible in the sets of $\mathscr{U}_u^{\text{in}}$.

We will study this problem without the assumption that W_{mec} is polyconvex in the first variable. We will prove that

$$\int_\Omega W^{qc}_{\text{mec}}(Du(x),\vec{n}(u(x)))\,dx+\int_{\operatorname{im}_{\mathrm{T}}(u,\Omega)}|D\vec{n}(y)|^2\,dy.$$

is the relaxation of the above problem in the sense of Theorem 3.5.3. We denote by W^{qc}_{mec} the quasiconvexification of W_{mec} in the first variable and we will assume that it is polyconvex.

The quasiconvexity has been an essential concept in the theory of lower semicontinuity and vectorial problems in the calculus of variations since the work of Morrey [117], see also [40], [121] and [141]. The computation of W^{qc} is difficult in general but has been done in some cases with high symmetry [32, 37, 46, 154]. The idea is construct test functions using lamination and rank-one convexity to prove the upper bound and then to show the resulting expression is polyconvex, which gives the lower bound.

To prove that the relaxation of the above functional is the quasiconvexification we will use the above-mentioned result of [16] to prove the lower semicontinuity. To prove the upper bound we will adapt the proof of [33], where they show that the relaxation of the energy

$$\int_\Omega W(Du)\,dx \quad \text{is} \quad \int_\Omega W^{qc}(Du)\,dx,$$

where $u\in W^{1,p}(\Omega)$, with $p\geqslant n$, and $\det Du>0$ a.e.

As a direct corollary of our result we obtain that the result of [33] can be extended to the functions in the class $\mathscr{A}_p$ (choosing W_{mec} such that $W_{\text{mec}}(F,\vec{n})=W(F)$ for all $F\in\mathbb{R}^{n\times n}$ and all $\vec{n}\in\mathbb{S}^{n-1}$).

The definition of the quasiconvexification of W is not trivial when W is infinite for some matrices. As in [33], we define the quasiconvex envelope $W^{qc}:\mathbb{R}^{n\times n}\to[0,\infty]$ of a Borel-measurable function $W:\mathbb{R}^{n\times n}\to[0,\infty]$ by

$$W^{qc}(F)=\inf\left\{\fint_{B(0,1)}W(D\varphi)\,dx:\varphi\in W^{1,\infty}(B(0,1),\mathbb{R}^n),\varphi(x)=Fx\text{ for }x\in\partial B(0,1)\right\}. \tag{3.5.1}$$

Now we present the coercivity and growth conditions of the elastic energy function W, which is linked to W_{mec} through the formula (3.5.7) below.

Let $\theta : (0,\infty) \to [0,\infty)$ be convex, $c > 0$. If

$$\theta(xy) \leqslant c(1+\theta(x))(1+\theta(y)) \text{ for all } x, y \in (0,\infty) \tag{3.5.2}$$

we say that θ satisfies the structure condition. We also suppose that θ satisfies

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty. \tag{3.5.3}$$

Let $W \in C(\mathbb{R}^{n\times n}_+; [0,\infty))$ and $p \geqslant 1$; we say that the growth condition of W holds if

$$\frac{1}{c}|F|^p + \frac{1}{c}\theta(\det F) - c \leqslant W(F) \leqslant c|F|^p + c\theta(\det F) + c. \tag{3.5.4}$$

We will also require the following conditions on W:

$$(\det F)^{1-q'} \lesssim W(F) \tag{3.5.5}$$

and there exists $h : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0} h(t) = 0$ and for all $F \in \mathbb{R}^{n\times n}_+$ and $\vec{n}, \vec{m} \in \mathbb{S}^{n-1}$ we have

$$|W_{\text{mec}}(F,\vec{n}) - W_{\text{mec}}(F,\vec{m})| \leqslant h(|\vec{n}-\vec{m}|)W_{\text{mec}}(F,\vec{n}). \tag{3.5.6}$$

Here, the mechanical response $W_{\text{mec}} : \mathbb{R}^{n\times n} \times \mathbb{S}^{n-1} \to [0,\infty)$ describes the coupling between the deformation and the director field through the formula

$$W_{\text{mec}}(F,\vec{n}) := W(V_{\vec{n}}^{-1}F) \quad \text{with } V_{\vec{n}} = \alpha\vec{n}\otimes\vec{n} + \alpha^{\frac{1}{n-1}}(I - \vec{n}\otimes\vec{n}) \tag{3.5.7}$$

for a fixed $\alpha > 0$. Observe that this tensor $V_{\vec{n}}$ is volume-preserving, i.e., $\det V_{\vec{n}} = 1$. Denote $C_\alpha = \max_{\vec{n}\in\mathbb{S}^{n-1}} |V_{\vec{n}}^{-1}|^p$, and note that $C_\alpha = \max\{\alpha^{-n}, \alpha^{-\frac{n}{n-1}}\}$.

Observe that (3.5.4) and (3.5.5) imply $\lim_{t\to 0}\theta(t) = \infty$.

Given $u_0 : \Gamma \to \mathbb{R}^n$, with Γ being an $(n-1)$-rectifiable set, define $\mathscr{B}$ as the set of $(u,\vec{n})$ where $u \in \mathscr{A}_p$ and, $u|_\Gamma = u_0$ and

$$\vec{n} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(u,\Omega), \mathbb{S}^{n-1}).$$

The energy functional $I : \mathscr{B} \to [0,\infty]$ that describes the nematic elastomer is the sum of two contributions,

$$I := I_{\text{nem}} + I_{\text{mec}}, \tag{3.5.8}$$

where $I_{\text{nem}} : \mathscr{B} \to [0,\infty)$ is defined as

$$I_{\text{nem}} := \int_{\mathrm{im}_{\mathrm{T}}(u,\Omega)} |D\vec{n}(y)|^2\,dy, \tag{3.5.9}$$

and $I_{\text{mec}} : \mathscr{B} \to [0,\infty]$ is defined by

$$I_{\text{mec}} := \int_\Omega W_{\text{mec}}(Du(x), \vec{n}(u(x)))\,dx. \tag{3.5.10}$$

Define

$$I^*_{\text{mec}} := \int_\Omega W^{qc}_{\text{mec}}(Du(x), \vec{n}(u(x)))\,dx \text{ and } I^* := I_{\text{nem}} + I^*_{\text{mec}}. \tag{3.5.11}$$

Now we state the main results of this section. We will prove them later in shorter results. We start with the upper bound.

Theorem 3.5.1. *Let $q > 1$ and $W \in C^0(\mathbb{R}^{n\times n}_+; [0,\infty))$ satisfy* (3.5.4) *and* (3.5.5) *for $p > n-1$, $\theta : (0,\infty) \to [0,\infty)$ convex satisfying* (3.5.2) *and* (3.5.3)*, and extend W to $\mathbb{R}^{n\times n}$ by $W(F) = \infty$ if $\det F \leqslant 0$. Let W^{qc} be defined as in* (3.5.1)*, W_{mec} defined as in* (3.5.7) *satisfying* (3.5.6)*, $\Omega \subset \mathbb{R}^n$ open bounded and Lipschitz. Then, for any $u \in \mathscr{A}_{p,q}(\Omega)$ and $\vec{n} \in W^{1,2}(\text{im}_{\text{T}}(u,\Omega), \mathbb{S}^{n-1})$ there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathscr{A}_{p,q}(\Omega)$ such that u_j converges weakly to u in $W^{1,p}$, $u_j - u \in W^{1,p}_0(\Omega,\mathbb{R}^n)$ for all j, $\chi_{\text{im}_{\text{T}}(u_j,\Omega)}\vec{n} \to \chi_{\text{im}_{\text{T}}(u,\Omega)}\vec{n}$ almost everywhere,*

$$\limsup_{j\to\infty} \int_{\text{im}_{\text{T}}(u_j,\Omega)} |D\vec{n}(y)|^2\,dy \leqslant \int_{\text{im}_{\text{T}}(u,\Omega)} |D\vec{n}(y)|^2\,dy$$

and

$$\limsup_{j\to\infty} \int_\Omega W_{\text{mec}}(Du_j(x), \vec{n}(u_j(x)))\,dx \leqslant \int_\Omega W^{qc}_{\text{mec}}(Du(x), \vec{n}(u(x)))\,dx.$$

Proof. It follows from Lemma 3.5.8 below. □

Existence of minimizers for I^* is given by the following result.

Theorem 3.5.2. *[16, Theorem 8.2.] Let Ω be a Lipschitz domain of $\mathbb{R}^n$, Γ an $(n-1)$-rectifiable subset of $\partial\Omega$ with $\mathscr{H}^{n-1}(\Gamma) > 0$, and $u_0 : \Gamma \to \mathbb{R}^n$. Let $p > n-1$, define $\mathscr{B}$ as the set $(u,\vec{n})$ where $u \in \mathscr{A}_p$, $u|_\Gamma = u_0$ and $\vec{n} \in W^{1,2}(\text{im}_{\text{T}}(u,\Omega), \mathbb{S}^{n-1})$. Let $W : \mathbb{R}^{n\times n}_+ \to [0,\infty)$ be a polyconvex function such that equations* (3.5.4) *and* (3.5.2) *for a constant $c > 0$ and a Borel function $\theta : (0,\infty) \to [0,\infty)$. Define I as in* (3.5.8)*–*(3.5.7)*. If $\mathscr{B} \neq \emptyset$ and I is not identically infinity in $\mathscr{B}$, then I attains its minimum in $\mathscr{B}$.*

The relaxation result is as follows.

Theorem 3.5.3. *Let $W \in C^0(\mathbb{R}^{n\times n}_+; [0,\infty))$ obey* (3.5.2), (3.5.4), (3.5.3) *with $p > n-1$,* (3.5.5) *with $q > 1$, such that W^{qc} is polyconvex and W_{mec} satisfies* (3.5.6)*. Let Ω be an open, bounded, Lipschitz, connected set, Γ an $(n-1)$-rectifiable set of $\partial\Omega$ with $\mathscr{H}^{n-1}(\Gamma) > 0$, $u_0 : \Gamma \to \mathbb{R}^n$ and $f \in C^0(\mathbb{R}^n)$ with $|f(t)| \lesssim (1+|t|^r)$ for some $r \in [1,p^*)$. Define $\mathscr{B}$ as the set of $(u,\vec{n})$ where $u \in \mathscr{A}_{p,q}(\Omega)$, $u|_\Gamma = u_0$ and $\vec{n} \in W^{1,2}(\text{im}_{\text{T}}(u,\Omega), \mathbb{S}^{n-1})$.*

We define the functionals $E, E^* : \mathscr{B} \to \mathbb{R} \cup \{\infty\}$ by

$$E[u,\vec{n}] = I(u,\vec{n}) + \int_\Omega f(u(x))\,dx \text{ and } E^*[u,\vec{n}] = I^*(u,\vec{n}) + \int_\Omega f(u(x))\,dx$$

Then the following assertions hold:

i) E^* is the relaxation of E in the sense that

$$E^*[u,\vec{n}] = \inf\{\liminf_{j\to\infty} E[u_j,\vec{n}_j] : (u_j\vec{n}_j) \in \mathscr{B}, u_j \rightharpoonup u \text{ in } W^{1,p}, \chi_{\text{im}_{\text{T}}(u_j,\Omega)}\vec{n}_j \to \chi_{\text{im}_{\text{T}}(u,\Omega)}\vec{n} \text{ a.e.}\};$$

ii) The functional E^* has a minimizer in the space $\mathscr{B}$.

Proof. By Theorem 3.5.1, for any $u \in \mathscr{A}_{p,q}(\Omega)$ and any $\vec{n} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(u,\Omega), \mathbb{S}^{n-1})$ there is $(u_j)_{j\in\mathbb{N}} \subset \mathscr{A}_{p,q}(\Omega)$ with $u_j = u$ on $\partial\Omega$, $u_j \to u$ in L^r, $u_j \rightharpoonup u$ in $W^{1,p}$, $\chi_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)}\vec{n} \to \chi_{\mathrm{im}_{\mathrm{T}}(u,\Omega)}\vec{n}$ a.e.,

$$\limsup_{j\to\infty} \int_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)} |D\vec{n}(y)|^2 dy \leqslant \int_{\mathrm{im}_{\mathrm{T}}(u,\Omega)} |D\vec{n}(y)|^2 dy$$

and

$$\limsup_{j\to\infty} \int_{\Omega} W_{\mathrm{mec}}(Du_j(x), \vec{n}(u_j(x)))dx \leqslant \int_{\Omega} W^{qc}_{\mathrm{mec}}(Du(x), \vec{n}(u(x)))dx.$$

Using $|f(t)| \lesssim (1+|t|^r)$, the Dominated Convergence theorem and that f is continuous we obtain

$$\int_{\Omega} f(u(x))dx = \lim_{j\to\infty}\int_{\Omega} f(u_j(x))dx.$$

Therefore we have

$$\limsup_{j\to\infty} E[u_j, \vec{n}] \leqslant E^*[u, \vec{n}].$$

This gives us the upper bound.

Let now $u \in \mathscr{A}_{p,q}(\Omega)$, $\vec{n} \in W^{1,2}(\mathrm{im}_{\mathrm{T}}(u,\Omega), \mathbb{S}^{n-1})$ and $(u_j, \vec{n}_j)_{j\in\mathbb{N}}$ be a sequence in $\mathscr{B}$ such that

$$u_j \rightharpoonup u \text{ in } W^{1,p}, \quad \chi_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)}\vec{n}_j \to \chi_{\mathrm{im}_{\mathrm{T}}(u,\Omega)}\vec{n} \text{ a.e.}$$

Therefore, Proposition 3.2.20 implies

$$\int_{\Omega} W^{qc}_{\mathrm{mec}}(Du, n\circ u)dx \leqslant \liminf_{j\to\infty}\int_{\Omega} W^{qc}_{\mathrm{mec}}(Du_j, n_j\circ u_j)dx \leqslant \liminf_{j\to\infty}\int_{\Omega} W_{\mathrm{mec}}(Du_j, n_j\circ u_j)dx,$$

and, as a consequence of Proposition 3.2.19, we get

$$\int_{\mathrm{im}_{\mathrm{T}}(u,\Omega)} |D\vec{n}(y)|^2 dy \leqslant \liminf_{j\to\infty}\int_{\mathrm{im}_{\mathrm{T}}(u_j,\Omega)} |D\vec{n}_j(y)|^2 dy.$$

Since $u_j \rightharpoonup u$ in $W^{1,p}$ we have $u_j \to u$ in L^r; using again $|f(t)| \lesssim (1+|t|^r)$, the Dominated Convergence theorem and that f is continuous we obtain

$$\int_{\Omega} f(u(x))dx = \lim_{j\to\infty}\int_{\Omega} f(u_j(x))dx. \tag{3.5.12}$$

Therefore we have obtained $E^*[u,\vec{n}] \leqslant \liminf_{j\to\infty} E^*[u_j, \vec{n}_j] \leqslant \liminf_{j\to\infty} E[u_j, \vec{n}_j]$, and the proof of part *i)* is finished.

Part *ii)* is Theorem 3.5.2 for $f = 0$. The proof under the inclusion of f is standard (see, e.g., [29, Theorem 7.7.-1], if necessary). □

Now we present the result under the incompresibility assumption. We start with the upper bound.

Theorem 3.5.4. *Let $W \in C^0(SL(n);[0,\infty))$ satisfy*

$$\frac{1}{c}|F|^p - c \leqslant W(F) \leqslant c|F|^p + c. \tag{3.5.13}$$

for $p > n-1$, $c > 0$ and extend W to $\mathbb{R}^{n\times n}$ by $W(F) = \infty$ if $\det F \neq 1$. Let W^{qc} be defined as in (3.5.1)*, W_{mec} defined as in* (3.5.7) *satisfying* (3.5.6) *and $\Omega \subset \mathbb{R}^n$ open bounded and Lipschitz. Then, for any $u \in \mathscr{A}^1_p(\Omega)$ and $\vec{n} \in W^{1,2}(\mathrm{im_T}(u,\Omega),\mathbb{S}^{n-1})$ there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathscr{A}^1_p(\Omega)$ which converges weakly to u in $W^{1,p}$, such that $u_j - u \in W^{1,p}_0(\Omega,\mathbb{R}^n)$ for all j, $\chi_{\mathrm{im_T}(u_j,\Omega)}\vec{n} \to \chi_{\mathrm{im_T}(u,\Omega)}\vec{n}$ almost everywhere,*

$$\limsup_{j\to\infty}\int_{\mathrm{im_T}(u_j,\Omega)} |D\vec{n}(y)|^2\,dy \leqslant \int_{\mathrm{im_T}(u,\Omega)} |D\vec{n}(y)|^2\,dy$$

and

$$\limsup_{j\to\infty}\int_\Omega W_{\mathrm{mec}}(Du_j(x),\vec{n}\circ u_j)(x)\,dx \leqslant \int_\Omega W^{qc}_{\mathrm{mec}}(Du(x),\vec{n}\circ u(x))\,dx.$$

Proof. It follows from Lemma 3.5.8 below. □

Theorem 3.5.5. *Let $W \in C^0(SL(n);[0,\infty))$ obey* (3.5.13) *with $p > n-1$ and such that W_{mec} satisfies* (3.5.6)*. Let Ω be an open, bounded, Lipschitz, connected set, Γ an $(n-1)$-rectifiable set of $\partial\Omega$ with $\mathscr{H}^{n-1}(\Gamma) > 0$, $u_0 : \Gamma \to \mathbb{R}^n$ and $f \in C^0(\mathbb{R}^n)$ with $|f(t)| \lesssim (1+|t|^r)$ for some $r \in [1,p^*)$. Define $\mathscr{B}'$ as the set of $(u,\vec{n})$ where $u \in \mathscr{A}^1_p(\Omega)$ and, $u|_\Gamma = u_0$ and $\vec{n} \in W^{1,2}(\mathrm{im_T}(u,\Omega),\mathbb{S}^{n-1})$.*

We define the functionals $E, E^* : \mathscr{B}' \to \mathbb{R}\cup\{\infty\}$ by

$$E[u,\vec{n}] = I(u,\vec{n}) + \int_\Omega f(u(x))\,dx \text{ and } E^*[u,\vec{n}] = I^*(u,\vec{n}) + \int_\Omega f(u(x))\,dx$$

for $u \in \mathscr{A}^1_p(\Omega)$. Finally assume that W^{qc} is polyconvex. Then the following assertions hold:

i) E^* is the relaxation of E in the sense that

$$E^*[u,\vec{n}] = \inf\{\liminf_{j\to\infty} E[u_j,\vec{n}_j] : (u_j,\vec{n}_j) \in \mathscr{B}', u_j \rightharpoonup u \text{ in } W^{1,p}, \chi_{\mathrm{im_T}(u_j,\Omega)}\vec{n}_j \to \chi_{\mathrm{im_T}(u,\Omega)}\vec{n} \text{ a.e.}\};$$

ii) The functional E^* has a minimizer in the space $\mathscr{B}'$.

Proof. The proof is the same as in Theorem 3.5.3. The only changes are that we use (3.5.13) instead of (3.5.4). See also [16, Remark 8.4.]. □

We now proceed with the proof, which is divided in several lemmas.

The product of two functions in L^1 is not in L^1 in general. However, the next lemma, whose proof can be found in [33], states that there are a lot of translations such that the product of the translated L^1 functions is in L^1. We will use this lemma to prove Lemma 3.5.7 below.

Lemma 3.5.6. *[33, Lemma 3.1] Let $\psi \in W^{1,\infty}(B(0,r),\overline{B(0,r)})$, $g \in L^1(B(0,r))$, $f \in L^1(B(x_0,2r))$ for some $x_0 \in \mathbb{R}^n$, $r > 0$. Then, there exists a measurable set $E \subset B(x_0,r)$ of positive measure with the following property. For any $a_0 \in E$, the function*

$$\tilde{f} = f(\psi(x-a_0)+a_0)g(x-a_0)$$

belongs to $L^1(B(a_0,r))$ with

$$\|\tilde{f}\|_{L^1(B(a_0,r))} \leqslant \frac{1}{|B(0,r)|}\|f\|_{L^1(B(x_0,2r))}\|g\|_{L^1(B(0,r))}.$$

Lemma 3.5.7. *Assume one of the following*

a) $W \in C^0(\mathbb{R}^{n\times n}_+;[0,\infty))$ satisfies the hypothesis of Theorem 3.5.1,

b) $W \in C^0(SL(n);[0,\infty))$ satisfies the hypothesis of Theorem 3.5.4,

and fix $F \in \mathbb{R}^{n\times n}_+$ in case a) *and $F \in SL(n)$ in case* b)*, $\vec{m} \in \mathbb{S}^{n-1}$ and $\eta \in (0,1)$. Then there is $\delta > 0$ such that for any $B = B(x_0,r)$, $\vec{n} \in W^{1,2}(\mathrm{im_T}(u,B),\mathbb{S}^{n-1})$ and*

$$u \in \begin{cases} \mathscr{A}_{p,q}(B) & \text{if } W \text{ satisfies a}),\\ \mathscr{A}^1_p(B) & \text{if } W \text{ satisfies b}),\end{cases}$$

with

$$\fint_B \left(|Du-F|^p + |\theta(\det Du)-\theta(\det F)| + |\vec{n}\circ u-\vec{m}|^p\right)dx \leqslant \delta \text{ in the case a}), \tag{3.5.14}$$

and

$$\fint_B \left(|Du-F|^p + |\vec{n}\circ u-\vec{m}|^p\right)dx \leqslant \delta \text{ in the case b}),$$

there exist $a_0 \in B\left(x_0,\frac{r}{2}\right)$ and

$$z \in \begin{cases} \mathscr{A}_{p,q}(B) & \text{if } W \text{ satisfies a}),\\ \mathscr{A}^1_p(B) & \text{if } W \text{ satisfies b}),\end{cases}$$

with $z = u$ on $B(x_0,r)\setminus B\left(a_0,\frac{r}{2}\right)$ and

$$\int_{B\left(a_0,\frac{r}{2}\right)} W_{\mathrm{mec}}(Dz,\vec{n}\circ z)\,dx \leqslant \int_{B\left(a_0,\frac{r}{2}\right)} (W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+\eta)\,dx. \tag{3.5.15}$$

Additionally,

$$\int_B |u-z|^p \leqslant cr^p \int_B (W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+1)\,dx. \tag{3.5.16}$$

If u is Lipschitz, then so is z.

Proof. This proof is partially based on that of [33, Lemma 3.2]. We will only prove the case a). The proof of case b) is analogous.

The L^p bound (3.5.16) follows from (3.5.4), (3.5.15) and Poincaré's inequality.

By the definition of quasiconvexification, (3.5.1), there exists $\varphi_\eta \in W^{1,\infty}\left(B\left(0,\frac{r}{2}\right)\right)$ such that $\varphi_\eta(x) = Fx$ on $\partial B\left(0,\frac{r}{2}\right)$, $\det D\varphi_\eta > 0$ a.e. and

$$\fint_{B(0\frac{r}{2})} W_{\mathrm{mec}}(D\varphi_\eta, \vec{m})\, dx \leqslant W^{qc}_{\mathrm{mec}}(F, \vec{m}) + \eta. \tag{3.5.17}$$

The function $F^{-1}\varphi_\eta$ is Lipschitz and is the identity in the border of $B\left(0,\frac{r}{2}\right)$. Therefore, thanks to [11, Theorem 1], $F^{-1}\varphi_\eta\left(B\left(0,\frac{r}{2}\right)\right) \subset \overline{B\left(0,\frac{r}{2}\right)}$. Moreover, $F^{-1}\varphi_\eta$ is invertible and its inverse is in $W^{1,1}$ (Theorem 3.4.2). Take $a_0 \in B\left(x_0,\frac{r}{2}\right)$ (to be chosen below), and set $v(x) = F^{-1}\varphi_\eta(x-a_0) + a_0$ and

$$z(x) = \begin{cases} u(v(x)) & \text{if } x \in B' = B\left(a_0,\frac{r}{2}\right), \\ u(x) & B(x_0,r)\setminus B', \end{cases}$$

It is clear that $z = u$ on $B(x_0,r)\setminus B'$, $\operatorname{im}_{\mathrm{T}}(v,B') = B'$ and $v^{-1} \in W^{1,1}(B',\mathbb{R}^n)$.

There exists a null set N such that for all $a_0 \notin N$ we have that $z \in W^{1,1}(B',\mathbb{R}^n)$, $\det Dz \in L^1(B)$ and $\operatorname{cof} Dz \in L^q$, see Lemmas 3.2.16 and 3.2.15. Moreover, since $v|_{\partial B'} = \mathrm{id}|_{\partial B'}$ we have $u\circ v|_{\partial B'} = u|_{\partial B'}$ and, hence, $z \in W^{1,1}(B,\mathbb{R}^n)$. Choose $a_0 \in E\setminus N$ using Lemma 3.5.6 applied to B' with $\psi = F^{-1}\varphi_\eta$, $f = |Du-F|^p + |\theta(\det Du) - \theta(\det F)|$ and $g = 1+\theta(\det(F^{-1}D\varphi_\eta))$. Then

$$\fint_{B'} (1+\theta(\det Dv))\left(|Du-F|^p + |\theta(\det Du)-\theta(\det F)|\right)\circ v\, dx \leqslant c_\eta\delta, \tag{3.5.18}$$

with c_η depending on η and F.

By (3.5.4) and (3.5.2) we have $\theta(\det D\varphi_\eta) \in L^1\left(B\left(0,\frac{r}{2}\right)\right)$ and $\theta(\det(F^{-1}D\varphi_\eta)) \in L^1\left(B\left(0,\frac{r}{2}\right)\right)$. Using (3.5.5) we also have $\int_{B\left(0,\frac{r}{2}\right)} \left(\frac{1}{\det D\varphi_\eta}\right)^{q'-1} dx < \infty$. Therefore, there exists $\gamma > 0$ (depending on F, $\vec{m}$ and η) such that

$$\int_{B\left(0,\frac{r}{2}\right)\cap\{\det D\varphi_\eta<\gamma\}} (1+\theta(\det(F^{-1}D\varphi_\eta)))\,dx \leqslant \frac{\left|B\left(0,\frac{r}{2}\right)\right|\eta}{\left(3+\|F^{-1}D\varphi_\eta\|^p_{L^\infty}\right)(1+|F|^p+\theta(\det F))} \tag{3.5.19}$$

and

$$\int_{B\left(0,\frac{r}{2}\right)\cap\{\det D\varphi_\eta<\gamma\}} (1+C_\alpha|D\varphi_\eta|^p + \theta(\det D\varphi_\eta))\,dx \leqslant \frac{1}{c}\left|B\left(0,\frac{r}{2}\right)\right|\eta, \tag{3.5.20}$$

where c is the constant of (3.5.4) and $C_\alpha = \max_{\vec{n}\in\mathbb{S}^{n-1}} |V^{-1}_{\vec{n}}|^p$.

Let $R_\eta = \|Dv\|_{L^\infty}$ and $M_\eta = \|D\varphi_\eta\|_{L^\infty}$. Since W is continuous in $\mathbb{R}^{n\times n}_+$ there is $\varepsilon > 0$ with $\varepsilon R_\eta C_\alpha \leqslant 1$ and $\varepsilon C_\alpha \leqslant 1$ such that

$$|W_{\mathrm{mec}}(\sigma,\vec{\ell}) - W_{\mathrm{mec}}(\zeta,\vec{k})| \leqslant \eta \tag{3.5.21}$$

for all $\sigma,\eta \in \mathbb{R}^{n\times n}_+$, with $|\zeta| \leqslant M_\eta$, $\det\zeta \geqslant \gamma$ and $|\sigma-\zeta| + |\vec{\ell}-\vec{k}| \leqslant \varepsilon R_\eta C_\alpha$.

In [60, Theorem 2.4 and Proposition 2.3] it is proved that W^{qc} is continuous in $\mathbb{R}^{n\times n}_+$. Hence, W^{qc}_{mec} is continuous and, consequently, there exists $\varepsilon > 0$ depending on η and F such that

(3.5.22)
$$|W^{qc}_{\mathrm{mec}}(\zeta,\vec{\ell}) - W^{qc}_{\mathrm{mec}}(F,\vec{\ell})| + |\theta(\det\zeta) - \theta(\det F)| \leqslant \eta \text{ for all } \zeta \text{ satisfying } |\zeta - F| \leqslant \varepsilon \text{ and } \vec{\ell} \in \mathbb{S}^{n-1}.$$

Now let $\rho > 0$, $\vec{\ell} \in \mathbb{S}^{n-1}$ and $\psi_{\vec{\ell}} \in W^{1,\infty}(B(0,1),\mathbb{R}^n)$ be such that $\psi_{\vec{\ell}}(x) = Fx$ on $\partial B(0,1)$ and

$$W^{qc}_{\mathrm{mec}}(F,\vec{\ell}) \geqslant \fint_{B(0,1)} W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{\ell})\,dx - \rho.$$

Define $M = \sup_{\vec{\ell}\in\mathbb{S}^{n-1}} \fint_{B(0,1)} W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{\ell})\,dx$. Then $M < \infty$ because

$$\fint_{B(0,1)} W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{\ell})\,dx \leqslant W^{qc}_{\mathrm{mec}}(F,\vec{\ell}) + \rho \leqslant W_{\mathrm{mec}}(F,\vec{\ell}) + \rho$$

and $\sup_{\vec{\ell}\in\mathbb{S}^{n-1}} W_{\mathrm{mec}}(F,\vec{\ell}) < \infty$ thanks to the continuity of W_{mec}. Using (3.5.6) we get that for all $\vec{\ell} \in \mathbb{S}^{n-1}$,

$$\begin{aligned}
W^{qc}_{\mathrm{mec}}(F,\vec{m}) - W^{qc}_{\mathrm{mec}}(F,\vec{\ell}) &\leqslant \fint_{B(0,1)} (W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{m}) - W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{\ell}))\,dx + \rho \\
&\leqslant h(|\vec{m}-\vec{\ell}|) \fint_{B(0,1)} W_{\mathrm{mec}}(D\psi_{\vec{\ell}},\vec{\ell})\,dx + \rho \\
&\leqslant h(|\vec{m}-\vec{\ell}|)M + \rho.
\end{aligned}$$

Analogously,
$$W^{qc}_{\mathrm{mec}}(F,\vec{\ell}) - W^{qc}_{\mathrm{mec}}(F,\vec{m}) \leqslant h(|\vec{m}-\vec{\ell}|)M + \rho.$$

Therefore, using (3.5.22) and $\lim_{t\to 0} h(t) = 0$, we have that there exists $\varepsilon > 0$ not depending on u, $\vec{n}$ or δ such that

$$|W^{qc}_{\mathrm{mec}}(\zeta,\vec{\ell}) - W^{qc}_{\mathrm{mec}}(F,\vec{m})| + |\theta(\det\zeta) - \theta(\det F)| \leqslant 2\eta \tag{3.5.23}$$

for all ζ and $\vec{\ell}$ satisfying $|\zeta - F| + |\vec{m}-\vec{\ell}| \leqslant \varepsilon$.

Set $\hat{\varphi}_\eta(x) = \varphi_\eta(x - a_0)$, and write

$$\int_{B'} \big(W_{\mathrm{mec}}(Dz,\vec{n}\circ z) - W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)\big)\,dx = I_1 + I_2 + I_3 + I_4,$$

with
$$I_1 = \int_{B'} \big(W_{\mathrm{mec}}(Dz,\vec{n}\circ z) - W_{\mathrm{mec}}(D\hat{\varphi}_\eta,\vec{n}\circ z)\big)\,dx,$$

$$I_2 = \int_{B'} \big(W_{\mathrm{mec}}(D\hat{\varphi}_\eta,\vec{n}\circ z) - W_{\mathrm{mec}}(D\hat{\varphi}_\eta,\vec{m})\big)\,dx,$$

$$I_3 = \int_{B'} \big(W_{\mathrm{mec}}(D\hat{\varphi}_\eta,\vec{m}) - W^{qc}_{\mathrm{mec}}(F,\vec{m})\big)\,dx$$

and

$$I_4 = \int_{B'} \left(W^{qc}_{\text{mec}}(F, \vec{m}) - W^{qc}_{\text{mec}}(Du, \vec{n} \circ u)\right) dx.$$

We will estimate these four integrals separately. Thanks to (3.5.17) we have $I_3 \leqslant \eta |B'|$. To estimate I_4 we use (3.5.23) to get

$$W^{qc}_{\text{mec}}(F, \vec{m}) \leqslant W^{qc}_{\text{mec}}(Du, \vec{n} \circ u) + \eta \quad \text{on the set where } |Du - F| + |\vec{m} - \vec{n} \circ u| \leqslant \varepsilon.$$

In $\{x \in B' : |Du(x) - F| + |\vec{m} - \vec{n} \circ u(x)| > \varepsilon\}$ we use (3.5.14) and Chebyshev's inequality to get

$$\begin{aligned} I_4 &\leqslant \eta |B'| + W^{qc}_{\text{mec}}(F, \vec{m}) |\{x \in B' : |Du(x) - F| + |\vec{m} - \vec{n} \circ u(x)| > \varepsilon\}| \\ &\leqslant \eta |B'| + W^{qc}_{\text{mec}}(F, \vec{m}) \frac{2^{p-1}}{\varepsilon^p} |B| \delta. \end{aligned}$$

To estimate I_2 we need to define the following sets

$$\omega = \{x \in B' : |\vec{n} \circ u(x) - \vec{m}| \geqslant \varepsilon R_\eta C_\alpha\}$$

and

$$\omega_d = \{x \in B' : \det D\hat{\varphi}_\eta(x) \geqslant \gamma\},$$

where ε and γ are those of (3.5.21). Doing the change of variables $z(x) = u(x')$, i.e., $x = v^{-1}(x')$, we obtain

$$I_2 = \int_{B'} \left(W_{\text{mec}}((D\hat{\varphi}_\eta) \circ v^{-1}(x'), \vec{n} \circ u(x')) - W_{\text{mec}}((D\hat{\varphi}_\eta) \circ v^{-1}(x'), \vec{m})\right) \det Dv^{-1}(x') \, dx',$$

and

$$⨍_{B'} \det Dv^{-1}(x') \, dx' = ⨍_{B'} \frac{1}{\det((Dv) \circ v^{-1}(x'))} \, dx' = ⨍_{B'} \frac{\det Dv(x)}{\det Dv(x)} \, dx = 1.$$

Using (3.5.4) and (3.5.21) we get

$$\begin{aligned} I_2 &\leqslant \int_{v(\omega_d) \setminus \omega} \eta \det Dv^{-1}(x') \, dx' + \int_{B' \setminus (v(\omega_d) \setminus \omega)} W_{\text{mec}}((D\hat{\varphi}_\eta) \circ v^{-1}(x'), \vec{n} \circ u(x')) \det Dv^{-1}(x') \, dx' \\ &\leqslant \eta |B'| + c \int_{B' \setminus (v(\omega_d) \setminus \omega)} \left(1 + |V^{-1}_{\vec{n} \circ u(x')}|^p |(D\hat{\varphi}_\eta) \circ v^{-1}(x')|^p + \theta(\det D\hat{\varphi}_\eta) \circ v^{-1}(x')\right) \det Dv^{-1}(x') \, dx'. \end{aligned}$$

Doing the change of variables $x = v^{-1}(x')$ and using (3.5.20) we obtain

$$\begin{aligned} &c \int_{B' \setminus v(\omega_d)} \left(1 + |V^{-1}_{\vec{n} \circ u(x')}|^p |(D\hat{\varphi}_\eta) \circ v^{-1}(x')|^p + \theta(\det D\hat{\varphi}_\eta) \circ v^{-1}(x')\right) \det Dv^{-1}(x') \, dx' \\ &\leqslant c \int_{B' \setminus \omega_d} \left(1 + C_\alpha |D\hat{\varphi}_\eta(x)|^p + \theta(\det D\hat{\varphi}_\eta(x))\right) dx \leqslant \eta |B'|. \end{aligned}$$

On the other hand, for $x \in \omega_d$ we have that $\det Dv(x) \geqslant \gamma \det F^{-1}$, so $\det Dv^{-1} \in L^\infty(v(\omega_d))$. Then, using (3.5.14), $\theta(\det D\hat{\varphi}_\eta) \in L^\infty(\omega_d)$ and Chebyshev's inequality we get

$$\begin{aligned} &c \int_{\omega \cap v(\omega_d)} \left(1 + |V^{-1}_{\vec{n} \circ u(x')}|^p |(D\hat{\varphi}_\eta) \circ v^{-1}(x')|^p + \theta(\det(D\hat{\varphi}_\eta)) \circ v^{-1}(x')\right) \det Dv^{-1}(x') \, dx' \\ &\lesssim |\omega| \lesssim \varepsilon^{-p} \delta |B'|, \end{aligned}$$

with the constant under $\lesssim$ depending on W, γ and η but not on δ, $\vec{n}$, u or z.

Hence, we have that there exists a constant $\tilde{c}$ depending on η and W but not on δ such that

$$I_2 \leqslant (2\eta + \tilde{c}\varepsilon^{-p}\delta)|B'|.$$

Next, we estimate I_1. Let

$$\omega' = \{x \in B' : |Du(x) - F| \circ v \geqslant \varepsilon C_\alpha\}.$$

Using that, in B',

$$Dz = (Du \circ v)Dv = [(Du - F) \circ v]\, Dv + D\hat{\varphi}_\eta$$

and that in $\omega_d \setminus \omega'$ we have $\det D\hat{\varphi}_\eta \geqslant \gamma$ and $|Du(x) - F| \circ v \leqslant \varepsilon C_\alpha$ we get

$$|Dz - D\hat{\varphi}_\eta| \leqslant [|Du - F| \circ v]\, |Dv| \leqslant \varepsilon R_\eta C_\alpha.$$

By (3.5.21) we have

$$\int_{\omega_d \setminus \omega'} \left(W_{\text{mec}}(Dz, \vec{n} \circ z) - W_{\text{mec}}(D\hat{\varphi}_\eta, \vec{n} \circ z)\right) dx \leqslant \eta |B'|.$$

Using the growth estimate (3.5.4) we obtain

$$W_{\text{mec}}(Dz, \vec{n} \circ z) \leqslant c\left(1 + C_\alpha \left[|Du|^p \circ v\right] |Dv|^p + \theta((\det Du) \circ v \det Dv)\right).$$

Hence using $|Dv| \leqslant R_\eta$ and (3.5.2) we get that, in B',

$$W_{\text{mec}}(Dz, \vec{n} \circ z) \leqslant cC_\alpha \left(1 + R_\eta^p |Du|^p \circ v + 1 + \theta((\det Du) \circ v)\right)(1 + \theta(\det Dv)). \tag{3.5.24}$$

To estimate the integral in ω' we observe that $|Du - F| \circ v \geqslant \varepsilon C_\alpha \geqslant \varepsilon$ implies

$$|Du| \circ v + 1 \leqslant |Du - F| \circ v + |F| + 1 \leqslant \left(\frac{|F| + 1}{\varepsilon} + 1\right) |Du - F| \circ v$$

and

$$\theta(\det Du) \circ v \leqslant |\theta(\det Du) \circ v - \theta(\det F)| + \frac{\theta(\det F)}{\varepsilon^p} |Du - F|^p \circ v.$$

Therefore, from (3.5.24) and (3.5.18) we obtain

$$\begin{aligned}
\int_{\omega'} W_{\text{mec}}(Dz, \vec{n} \circ z) &\leqslant cC_\alpha \int_{\omega'} (1 + \theta(\det Dv(x)))\left(2 + R_\eta^p |Du|^p + \theta(\det Du)\right) \circ v(x)\, dx \\
&\leqslant c' \int_{\omega'} (1 + \theta(\det Dv(x)))\left(|Du - F|^p + |\theta(\det Du) - \theta(\det F)|\right) \circ v(x)\, dx \\
&\leqslant c'_\eta \delta |B'|.
\end{aligned}$$

The constant c'_η depends on W, η and F but not on δ. In $B' \setminus (\omega_d \cup \omega')$ we have $|Du - F| \circ v \leqslant \varepsilon C_\alpha \leqslant 1$ and $\det D\hat{\varphi}_\eta < \gamma$. Then we have $|Du| \circ v \leqslant |F| + 1$ and thanks to (3.5.22) we also obtain $\theta(\det Du) \circ v \leqslant \theta(\det F) + 1$. Therefore (3.5.24) implies

$$\begin{aligned}
W_{\text{mec}}(Dz, \vec{n} \circ z) &\leqslant cC_\alpha \left(3 + R_\eta^p (1 + |F|)^p + \theta(\det F)\right)(1 + \theta(\det Dv)) \\
&\leqslant c_* \left(3 + \|F^{-1} D\varphi_\eta\|_{L^\infty}^p\right)\left(1 + |F|^p + \theta(\det F)\right)(1 + \theta(\det Dv)),
\end{aligned}$$

with c_* depending only on W. Hence, thanks to (3.5.19) we get

$$\int_{B'\setminus(\omega'\cup\omega_d)} W_{\mathrm{mec}}(Dz,\vec{n}\circ z)\,dx \leqslant c_*\eta|B'|.$$

Hence

$$I_1 \leqslant (\eta + c'_\eta\delta + c_*\eta)|B'|.$$

Adding the estimates for I_1, I_2, I_3 and I_4 we obtain

$$\begin{aligned}&\int_{B'}\left(W_{\mathrm{mec}}(Dz,\vec{n}\circ z) - W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)\right)dx\\ &\quad\leqslant\left(\eta + c'_\eta\delta + c_*\eta + 2\eta + \frac{\tilde{c}}{\varepsilon^p}\delta + \eta + \eta + W^{qc}(F,\vec{m})\frac{2^{p-1}}{\varepsilon^p}\delta\right)|B'|.\end{aligned}$$

Recall that η, c'_η, c_*, $\tilde{c}$ and ε do not depend on δ. Then, choosing δ small enough, we have (3.5.15). Using the growth condition (3.5.4) we obtain $Dz \in L^p(B)$, so $z \in W^{1,p}(B)$.

Recall that a_0 was chosen so that $\det Dz \in L^1(B)$ and $\operatorname{cof} Dz \in L^q(B)$. Then Lemma 3.3.4 gives

$$z \in \mathscr{A}_{p,q}(B)$$

and the proof is completed. □

In the following lemma we apply Lemma 3.5.7 in the Lebesgue points of Du and $\vec{n}\circ u$. The proof is based on that of [33, Lemma 3.3.].

Lemma 3.5.8. *Let $\Omega\subset\mathbb{R}^n$ open, Lipschitz and bounded, and assume* a) *or* b) *of Lemma 3.5.7. Then for any*

$$u \in \begin{cases}\mathscr{A}_{p,q}(\Omega) & \text{if } W \text{ satisfies a)},\\ \mathscr{A}^1_p(\Omega) & \text{if } W \text{ satisfies b)},\end{cases}$$

there is a sequence

$$u_j \in \begin{cases}\mathscr{A}_{p,q}(\Omega) & \text{if } W \text{ satisfies a)},\\ \mathscr{A}^1_p(\Omega) & \text{if } W \text{ satisfies b)},\end{cases}$$

such that $u_j \rightharpoonup u$ in $W^{1,p}$, $u_j = u$ on $\partial\Omega$ and for any $\vec{n}\in W^{1,2}(\mathrm{im_T}(u,\Omega),\mathbb{S}^{n-1})$ we have $\chi_{\mathrm{im_T}(u_j,\Omega)}\vec{n} \to \chi_{\mathrm{im_T}(u,\Omega)}\vec{n}$ almost everywhere,

$$\limsup_{j\to\infty}\int_{\mathrm{im_T}(u_j,\Omega)}|D\vec{n}(y)|^2\,dy \leqslant \int_{\mathrm{im_T}(u,\Omega)}|D\vec{n}(y)|^2\,dy$$

and

$$\limsup_{j\to\infty}\int_\Omega W_{\mathrm{mec}}(Du_j,\vec{n}(u_j))\,dx \leqslant \int_\Omega W^{qc}_{\mathrm{mec}}(Du,\vec{n}(u))\,dx.$$

If, additionally, $u\in W^{1,\infty}(\Omega,\mathbb{R}^n)$, then we can take $u_j\in W^{1,\infty}(\Omega,\mathbb{R}^n)$.

Proof. We will only prove the case a), the proof of case b) being completely analogous. Fix $\eta \in (0,1)$. It is enough to construct

$$w \in \begin{cases} \mathscr{A}_{p,q}(\Omega) & \text{if } W \text{ satisfies } a), \\ \mathscr{A}^1_p(\Omega) & \text{if } W \text{ satisfies } b), \end{cases}$$

such that $\|u-w\|_{L^p} \leqslant \eta$, $w = u$ on $\partial\Omega$, $\mathrm{im_T}(w,\Omega) = \mathrm{im_T}(u,\Omega)$ and

$$\int_\Omega W_{\mathrm{mec}}(Dw, \vec{n}\circ w)\,dx \leqslant \int_\Omega W^{qc}_{\mathrm{mec}}(Du, \vec{n}\circ u)\,dx + \eta. \tag{3.5.25}$$

Indeed, we can construct u_j as the w of the statement corresponding to $\eta = j^{-1}$. Then $u_j \to u$ in L^p and, thanks to (3.5.25), we will have $\|u_j\|_{W^{1,p}(\Omega)}$ uniformly bounded, so $u_j \rightharpoonup u$ in $W^{1,p}$.

In the case that $\int_\Omega W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)\,dx = \infty$, we can take $w = u$. So we will assume $W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u) \in L^1(\Omega)$. Using the convexity of θ we have

$$\frac{1}{cC_\alpha}|F|^p + \frac{1}{c}\theta(\det F) - c \leqslant W^{qc}_{\mathrm{mec}}(F,\vec{m}) \text{ for all } F \in \mathbb{R}^{n\times n}_+ \text{ and } \vec{m} \in \mathbb{S}^{n-1},$$

where c is the constant of (3.5.4) and $C_\alpha = \max_{\vec{m}\in\mathbb{S}^{n-1}} |V^{-1}_{\vec{m}}|^p$. This is because the left-hand side of the inequality above is polyconvex, hence quasiconvex. Hence, $|Du|^p$ and $\theta(\det Du)$ are integrable. On the other hand, we have $\vec{n}\circ u \in L^\infty(\Omega)$. Denote by E the intersection of the set of p-Lebesgue points of Du and $\vec{n}\circ u$ and Lebesgue points of $\theta(\det Du)$. Given $x \in E$, let $F_x = Du(x)$ and $\vec{m}_x = \vec{n}\circ u(x)$, and choose δ_x as in Lemma 3.5.7 for this F_x, $\vec{m}_x$ and η as above.

We will construct a sequence of $\{(w_j,\Omega_j)\}_{j\in\mathbb{N}}$ such that $w_j \in \mathscr{A}_{p,q}(\Omega)$, $\{\Omega_j\}_{j\in\mathbb{N}}$ is a decreasing sequence of open subsets of Ω, $w_j = u$ on Ω_j and $\mathrm{im_T}(w_j,\Omega) = \mathrm{im_T}(u,\Omega)$. Set $w_0 = u$ and $\Omega_0 = \Omega$. The passage from (w_j,Ω_j) to (w_{j+1},Ω_{j+1}) is as follows. For all $x \in E\cap\Omega_j$ we choose $r_j(x) \in (0,\eta)$ such that $B(x,r_j(x)) \subset \Omega_j$, $B(x,r_j(x)) \in \mathscr{U}_u$ and

$$⨍_{B(x,r)} \left(|Dw_j(x') - F_x|^p + |\theta(\det Dw_j(x')) - \theta(\det F_x)| + |\vec{n}\circ w_j(x') - \vec{m}_x|^p\right) dx' \leqslant \delta_x$$

for all $r < r_j(x)$. The union of this collection of balls $B(x,r_j(x))$ covers Ω_j up to a set of measure zero. Extract a finite disjoint subcover $\{B(x_k,r_k)\}_{k=0}^M$ such that

$$\left|\bigcup_{k=0}^M B(x_k,r_k)\right| \geqslant \frac{1}{2}|\Omega_j|.$$

Define w_{j+1} as w_j on $\Omega\setminus\bigcup_{k=0}^M B(x_k,r_k)$ and as the function z of Lemma 3.5.7 in each of the balls $B(x_k,r_k)$. Then $w_{j+1} = w_j = u$ on $\partial\Omega$ and thanks to Lemma 3.3.2, we get

$$w_{j+1} \in \begin{cases} \mathscr{A}_{p,q}(\Omega) & \text{if } W \text{ satisfies } a), \\ \mathscr{A}^1_p(\Omega) & \text{if } W \text{ satisfies } b). \end{cases}$$

Let $B\left(x'_k, \frac{r_k}{2}\right) \subset B(x_k,r_k)$ be the balls given by Lemma 3.5.7 and take $\{U_i\}_{i\in\mathbb{N}} \in \mathscr{U}_{w_j}\cap\mathscr{U}_{w_{j+1}}$ such that $U_i \subset U_{i+1}$ and $\bigcup_{i\in\mathbb{N}} U_i = \Omega$. Then, using the definition of $\mathrm{im_T}(w_j,\Omega)$ (Definition 3.2.10) and that $w_j, w_{j+1} \in \mathscr{A}_{p,q}(\Omega)$ we have

$$\mathrm{im_T}(w_j,\Omega) = \bigcup_{i\in\mathbb{N}} \mathrm{im_T}(w_j,U_i) \quad \text{and} \quad \mathrm{im_T}(w_{j+1},\Omega) = \bigcup_{i\in\mathbb{N}} \mathrm{im_T}(w_{j+1},U_i).$$

Let $y \in \mathrm{im}_{\mathrm{T}}(w_j,\Omega)$, then, there exists i_0 such that $y \in \mathrm{im}_{\mathrm{T}}(w_j,U_{i_0})$ and $\bigcup_{k=0}^{M} \overline{B}\left(x'_k,\frac{r_k}{2}\right) \subset U_{i_0}$. Therefore $w_{j+1} = w_j$ on ∂U_{i_0}, so $y \in \mathrm{im}_{\mathrm{T}}(w_{j+1},U_{i_0}) \subset \mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega)$. Hence

$$\mathrm{im}_{\mathrm{T}}(w_j,\Omega) \subset \mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega).$$

Doing the same argument starting on $\mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega)$ we obtain $\mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega) \subset \mathrm{im}_{\mathrm{T}}(w_j,\Omega)$, so $\mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega) = \mathrm{im}_{\mathrm{T}}(w_j,\Omega)$, and, by induction $\mathrm{im}_{\mathrm{T}}(w_{j+1},\Omega) = \mathrm{im}_{\mathrm{T}}(u,\Omega)$.

By Lemma 3.5.7, the balls $B\left(x'_k,\frac{r_k}{2}\right) \subset B(x_k,r_k)$ satisfy

$$\int_{B\left(x'_k,\frac{r_k}{2}\right)} W_{\mathrm{mec}}(Dw_{j+1},\vec{n}\circ w_{j+1})\,dx \leqslant \int_{B\left(x'_k,\frac{r_k}{2}\right)} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+\eta\right)dx \tag{3.5.26}$$

and

$$\int_{B(x_k,r_k)} |w_{j+1}-u|^p\,dx \leqslant c\eta^p \int_{B(x_k,r_k)} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+1\right)dx. \tag{3.5.27}$$

Set $\Omega_{j+1} = \Omega_j \setminus \bigcup_{k=0}^{M} \overline{B}\left(x'_k,\frac{r_k}{2}\right)$. It is clear that $w_{j+1} = w_j = u$ on Ω_{j+1} and that $|\Omega_{j+1}| \leqslant (1-2^{-n-1})|\Omega_j|$. Hence, the construction of w_{j+1} is completed and we only have to prove that for j big enough, w_j has the desired properties. Thanks to (3.5.27) we have

$$\int_{\Omega} |w_j-u|^p\,dx \leqslant c\eta^p \int_{\Omega} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+1\right)dx$$

so w_j is close to u in L^p, independently of j.

From (3.5.26) we obtain

$$\int_{\Omega\setminus\Omega_j} W_{\mathrm{mec}}(Dw_{j+1},\vec{n}\circ w_{j+1})\,dx \leqslant \int_{\Omega\setminus\Omega_j} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+\eta\right)dx,$$

which implies

$$\int_{\Omega} W_{\mathrm{mec}}(Dw_{j+1},\vec{n}\circ w_{j+1})\,dx \leqslant \int_{\Omega\setminus\Omega_j} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+\eta\right)dx + \int_{\Omega_j} W_{\mathrm{mec}}(Du,\vec{n}\circ u)\,dx.$$

Using $|\Omega_j| \leqslant (1-2^{-n-1})^j|\Omega| \to 0$ and that thanks to (3.5.4) we have $W_{\mathrm{mec}}(Du,\vec{n}\circ u) \in L^1(\Omega)$, for j large enough we get

$$\int_{\Omega} W_{\mathrm{mec}}(Dw_{j+1},\vec{n}\circ w_{j+1})\,dx \leqslant \int_{\Omega} \left(W^{qc}_{\mathrm{mec}}(Du,\vec{n}\circ u)+2\eta\right)dx$$

and the proof is concluded. □

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